

Olympiad Problems Set -03

20 Selected Problems with Answer Key

Questions

- Let a and b be two integers. Suppose $x^2 - x - 1$ is a factor of $ax^5 + bx^4 + 1$. Find the value of a .
- The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, p is zero. What is the value of $\frac{n}{p}$?
- For integers $a_1, \dots, a_n \in \{1, 2, \dots, 9\}$, we use $\overline{a_1 a_2 \dots a_n}$ to denote the number $10^{n-1}a_1 + 10^{n-2}a_2 + \dots + 10a_{n-1} + a_n$. Given that $\overline{ab} = b^2$ and $\overline{acbc} = (\overline{ba})^2$. Find $a + b + c$.
- The sum $\frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{14 \times 15 \times 16}$ is $\frac{m}{n}$ in lowest terms. Find the value of $n - 5m$.
- How many real numbers x satisfy $3^{2x+2} - 3^{x+3} - 3^x + 3 = 0$?
- Find the sum of all possible values of a such that the following equation has real root in x : $(x - a)^2 + (x^2 - 3x + 2)^2 = 0$.
- Find a natural number n such that $2^8 + 2^{10} + 2^n$ is a perfect square number.
- Given that the real numbers s, t satisfy $19s^2 + 99s + 1 = 0$, $t^2 + 99t + 19 = 0$, and $st \neq 1$. Find the value of $\frac{st + 4s + 1}{t}$.
- Given $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 0$ and $\frac{1}{x} - \frac{6}{y} - \frac{5}{z} = 0$. Find the value of $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$.
- Find the value of the smallest positive integer m such that the equation $x^2 + 2(m + 5)x + (100m + 9) = 0$ has only integer solutions.
- Given that $f(x)$ is a polynomial of degree 3, and its remainders are $2x - 5$ and $-3x + 4$ when divided by $x^2 - 1$ and $x^2 - 4$ respectively. Find $f(-3)$.
- The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1$ with the condition $a < b < c < d$ is
- Given that $f(x) = x^2 + ax + b$ is a polynomial with integral coefficients. If f is a common factor of polynomials $g(x) = x^4 - 3x^3 + 2x^2 - 3x + 1$ and $h(x) = 3x^4 - 9x^3 + 2x^2 + 3x - 1$, find $f(x)$.
- Suppose f is a function satisfying $f(x + x^{-1}) = x^6 + x^{-6}$, for all $x \neq 0$. Determine $400 - f(3)$.
- Suppose x_1, x_2, x_3 are roots of $(11 - x)^3 + (13 - x)^3 = (24 - 2x)^3$. What is $x_1 + x_2 + x_3$?
- Find the value of $\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$ when $x = \sqrt{19 - 8\sqrt{3}}$.

17. What is the negative of the sum of the reciprocals of the roots of $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?
18. If n is the largest positive integer such that $\sqrt{n-100} + \sqrt{n+100}$ is a rational number, find the value of $\sqrt{n-1}$.
19. Factorize $x^4 + y^4 + (x+y)^4$.
20. How many ordered pairs of integers (x, y) satisfy $x^2 + y^2 = 2(x+y) + xy$?

Answer Key

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|--------------|---------------------------|
| 1. $a = 3$ | 11. 53 |
| 2. 8 | 12. 6 |
| 3. 18 | 13. $f(x) = x^2 - 3x + 1$ |
| 4. 95 | 14. 78 |
| 5. 2 | 15. 36 |
| 6. 3 | 16. 5 |
| 7. $n = 10$ | 17. 1 |
| 8. -5 | 18. 50 |
| 9. -1 | 19. $2(x^2 + xy + y^2)^2$ |
| 10. $m = 90$ | 20. 6 |