LOGARITHMS & EXPONENTIAL FUNCTIONS

The Complete Module

Theory · Graphs · Log Tables · Equations & Inequalities
Nested Logarithms · Domain–Range · Limits, Continuity & Differentiability

For **JEE Mains**, **JEE Advanced** & **Mathematical Olympiads**

Simplified Education

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1. The Exponential Function

1.1 Definition and Basic Idea

For a fixed real number $a > 0$, $a \neq 1$, the **exponential function** with base a is

$$f(x) = a^x, \qquad x \in \mathbb{R}.$$

The restrictions on a are not decorative — they are forced by mathematics itself:

- If $a < 0$, then $a^{1/2} = \sqrt{a}$ is not real; the expression a^x cannot be defined for all real x.
- If $a = 0$, then $0^x = 0$ for $x > 0$ but 0^x is undefined for $x \leq 0$.
- If $a = 1$, then $1^x = 1$ is a constant function — it is well defined but useless as an "exponential", and (crucially) it has no inverse.

How is a^x defined for *irrational* x, say $2^{\sqrt{2}}$? By a limiting process: take rational approximations $1.4,\ 1.41,\ 1.414, \ldots$ of $\sqrt{2}$ and define

$$2^{\sqrt{2}} = \lim_{n\to\infty} 2^{r_n}, \qquad r_n \to \sqrt{2},\ r_n \in \mathbb{Q}.$$

This is why a^x turns out to be a *continuous* function on $\mathbb{R}$ — continuity is built into its very definition.

Properties of a^x

For $a, b > 0$ and $x, y \in \mathbb{R}$:

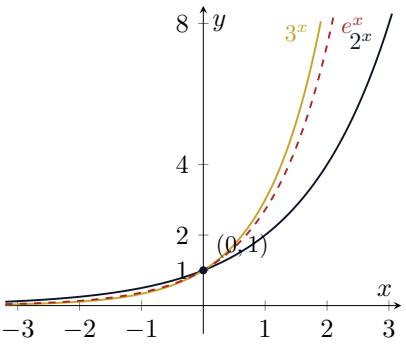
$$a^x\, a^y = a^{x+y}, \qquad \frac{a^x}{a^y} = a^{x-y}, \qquad (a^x)^y = a^{xy}, \qquad (ab)^x = a^x b^x, \qquad a^{-x} = \frac{1}{a^x}, \qquad a^0 = 1.$$

Moreover $a^x > 0$ for *every* real x: an exponential is never zero and never negative.

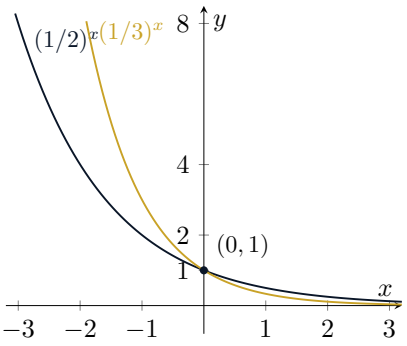
1.2 Graph of $y = a^x$

Two fundamentally different shapes occur, depending on whether $a > 1$ or $0 < a < 1$.

Case $a > 1$ (increasing)



Case $0 < a < 1$ (decreasing)



Anatomy of the graph of $y = a^x$

Feature	$a > 1$	$0 < a < 1$
Domain	$\mathbb{R}$	$\mathbb{R}$
Range	$(0, \infty)$	$(0, \infty)$
Monotonicity	strictly increasing	strictly decreasing
Fixed point on graph	$(0,1)$ and $(1,a)$	$(0,1)$ and $(1,a)$
$x \to +\infty$	$a^x \to \infty$	$a^x \to 0^+$
$x \to -\infty$	$a^x \to 0^+$	$a^x \to \infty$
Asymptote	$y = 0$ (as $x \to -\infty$)	$y = 0$ (as $x \to +\infty$)
Injective (one–one)?	Yes	Yes
Convexity	convex (bends upward)	convex (bends upward)

Every exponential graph passes through $(0,1)$, hugs the x-axis on one side, and explodes on the other. Since a^x is one–one and onto $(0, \infty)$, it has an inverse — *that inverse is the logarithm.*

Comparing exponential curves at a glance

For $x > 0$: the *larger* the base, the *higher* the curve ($3^x > e^x > 2^x$). For $x < 0$ this order *reverses* ($3^x < e^x < 2^x$). All curves pinch through $(0,1)$. Also note the reflection symmetry: $y = (1/a)^x$ is the mirror image of $y = a^x$ in the y-axis, because $(1/a)^x = a^{-x}$.

1.3 The Number e

Among all bases, one is singled out by calculus: the base for which the curve $y = a^x$ has slope exactly 1 at the point $(0,1)$. That base is

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.718281828\ldots$$

e is irrational (in fact transcendental). Writing $\ln x := \log_e x$, the pair e^x and $\ln x$ have the cleanest calculus of all exponential/log functions, which is why every limit and derivative formula in Section 9 onwards is stated first for base e.

Why $2 < e < 3$? A one-line Olympiad argument

$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \le 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 3$, while the first two terms already give ≥ 2. The sequence is increasing and bounded, hence convergent — its limit is defined to be e.

2. The Logarithm: Definition and First Principles

2.1 Definition

Definition of logarithm

Let $a > 0$, $a \neq 1$ and $N > 0$. Then

$$\boxed{\log_a N = x \iff a^x = N}$$

$\log_a N$ is *the power to which the base a must be raised to produce N*. The three conditions to memorise: **base positive, base $\neq 1$, argument positive.**

Read $\log_2 8 = 3$ as: "2 raised to what gives 8? Answer 3." Every logarithmic statement is an exponential statement in disguise:

$$\log_{10} 1000 = 3 \;\Leftrightarrow\; 10^3 = 1000, \qquad \log_5 \frac{1}{25} = -2 \;\Leftrightarrow\; 5^{-2} = \tfrac{1}{25}, \qquad \log_9 27 = \tfrac{3}{2} \;\Leftrightarrow\; 9^{3/2} = 27.$$

Immediate consequences of the definition (derive them, do not just memorise):

$$\log_a 1 = 0, \qquad \log_a a = 1, \qquad a^{\log_a N} = N, \qquad \log_a a^x = x.$$

The last two are the **fundamental identities**: they say that a^x and $\log_a x$ undo each other — they are inverse functions.

Common Mistake

$\log_a N$ is *undefined* when $N \leq 0$. Statements like $\log_2(-4) =?$ have no meaning in real mathematics. Also $\log_1 N$ is undefined: $1^x = 1$ always, so $1^x = 5$ has no solution and $1^x = 1$ has infinitely many. Finally, $a^{\log_a N} = N$ requires $N > 0$; do not "cancel" blindly. E.g. $2^{\log_2(x^2)} = x^2$ is valid for $x \neq 0$, but $2^{\log_2 x} = x$ needs $x > 0$.

2.2 Standard Bases and Notation

Notation	Meaning	Where used
$\log x$ or $\lg x$	$\log_{10} x$ (*common* logarithm)	log tables, orders of magnitude, pH, dB
$\ln x$	$\log_e x$ (*natural* logarithm)	all of calculus
$\log_2 x$	binary logarithm	computer science, information theory

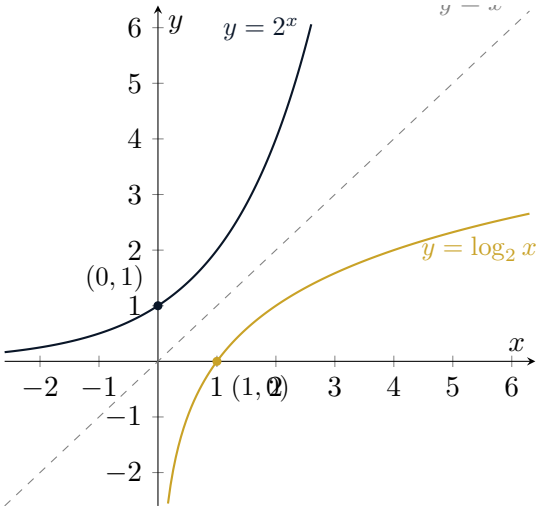
Convention alert

In JEE papers, $\log x$ with no base printed usually means $\log_{10} x$ in numerical/table contexts, but in calculus chapters many books write $\log x$ for $\ln x$. Always let the context (or the paper's instruction) decide; in this module $\log x$ means $\log_{10} x$ and $\ln x$ means $\log_e x$.

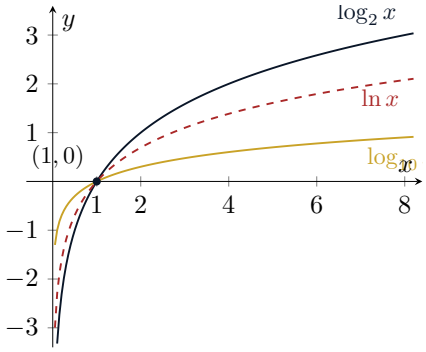
2.3 Graph of $y = \log_a x$

Since $\log_a x$ is the inverse of a^x, its graph is the **reflection of $y = a^x$ in the line $y = x$.**

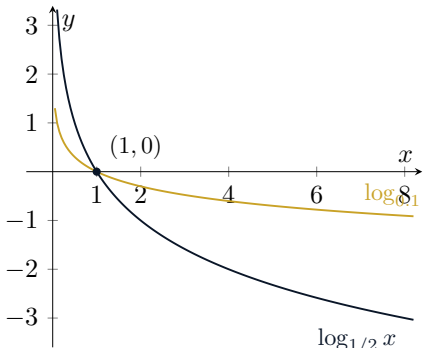
$y = 2^x$ and $y = \log_2 x$: mirror images in $y = x$



Base > 1: increasing



Base in $(0, 1)$: decreasing



Anatomy of the graph of $y = \log_a x$

Feature	$a > 1$	$0 < a < 1$
Domain	$(0, \infty)$	$(0, \infty)$
Range	$\mathbb{R}$	$\mathbb{R}$
Monotonicity	strictly increasing	strictly decreasing
Passes through	$(1, 0)$ and $(a, 1)$	$(1, 0)$ and $(a, 1)$
$x \to 0^+$	$\log_a x \to -\infty$	$\log_a x \to +\infty$
$x \to \infty$	$\log_a x \to +\infty$	$\log_a x \to -\infty$
Asymptote	$x = 0$ (the y-axis)	$x = 0$ (the y-axis)
Sign	< 0 on $(0, 1)$, > 0 on $(1, \infty)$	> 0 on $(0, 1)$, < 0 on $(1, \infty)$
Concavity	concave (bends downward)	convex

The sign chart you will use 100 times

$\log_a x > 0 \iff$ (a and x are on the *same side* of 1), i.e. both > 1 or both in $(0, 1)$.
$\log_a x < 0 \iff$ (a and x are on *opposite sides* of 1).
Example: $\log_{1/2} 5 < 0$ instantly; $\log_{0.3} 0.04 > 0$ instantly. This one observation solves most "compare with zero" questions in seconds.

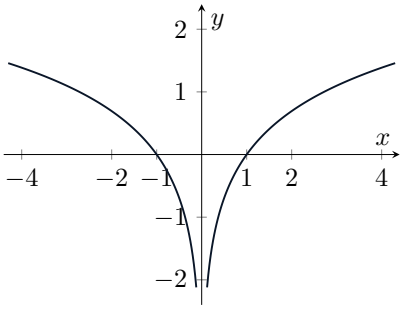
Comparing log curves for $x > 1$

For $x > 1$, a *bigger base means a smaller (flatter) logarithm*: $\log_{10} x < \ln x < \log_2 x$. For $0 < x < 1$ the order flips. To remember: at $x =$ base, the curve reaches height 1; a big base takes longer to reach height 1, so it rises more slowly.

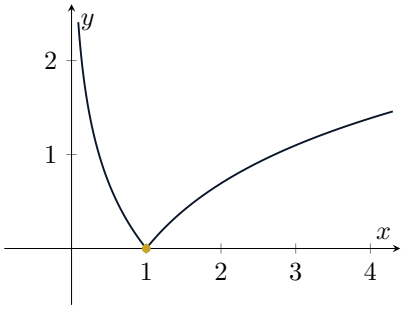
2.4 Graph Transformations Worth Knowing

- $y = \ln|x|$: even extension; two branches, domain $\mathbb{R} \setminus \{0\}$.
- $y = |\ln x|$: the part below the x-axis (i.e. $0 < x < 1$) is folded up.
- $y = \ln \frac{1}{x} = -\ln x$: reflection in the x-axis.
- $y = \ln(x-h)$: shift right by h; vertical asymptote moves to $x = h$.
- $y = \log_a x$ vs $y = \log_{1/a} x$: reflections of each other in the x-axis, since $\log_{1/a} x = -\log_a x$.
- $y = \ln x^2 = 2\ln|x|$ — note the domain enlarges to $x \neq 0$; it is *not* the same function as $2\ln x$.

$y = \ln|x|$



$y = |\ln x|$ — **corner at** $(1, 0)$



The corner of $y = |\ln x|$ at $x = 1$ is exactly where differentiability fails — we return to this in Section 10.

3. Laws of Logarithms

Throughout, all bases are positive and $\neq 1$, and all arguments are positive.

The fundamental laws

(L1) Product law: $\log_a(MN) = \log_a M + \log_a N$

(L2) Quotient law: $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(L3) Power law: $\log_a(M^p) = p \log_a M$

(L4) Base-power law: $\log_{a^q} M = \dfrac{1}{q}\log_a M \implies \log_{a^q} M^p = \dfrac{p}{q}\log_a M$

(L5) Change of base: $\log_a M = \dfrac{\log_b M}{\log_b a}$ (any valid base b)

(L6) Reciprocal (base swap): $\log_a b = \dfrac{1}{\log_b a} \implies \log_a b \cdot \log_b a = 1$

(L7) Chain rule of logs: $\log_a b \cdot \log_b c = \log_a c, \qquad \log_a b \cdot \log_b c \cdot \log_c d = \log_a d$

(L8) Fundamental identities: $a^{\log_a N} = N, \qquad \log_a a^x = x$

(L9) The star identity: $\boxed{a^{\log_b c} = c^{\log_b a}}$

Where do these come from? Every law is an exponent law wearing a log costume. Proof of (L1): let $\log_a M = x$, $\log_a N = y$. Then $M = a^x$, $N = a^y$, so $MN = a^{x+y}$, i.e. $\log_a(MN) = x + y$. ■

Proof of (L9): take $\log_b$ of both sides. LHS gives $\log_b c \cdot \log_b a$; RHS gives $\log_b a \cdot \log_b c$. Equal — done. This innocent-looking identity kills many JEE problems (e.g. it instantly tells you $2^{\log_{10} 5} = 5^{\log_{10} 2}$).

Common Mistake

- $\log_a(M+N) \neq \log_a M + \log_a N$. The product law is about *products*, not sums. There is **no** formula for $\log(M+N)$.
- $(\log_a M)^p \neq p \log_a M$. Note carefully: $\log^2 x$ conventionally means $(\log x)^2$, *not* $\log(\log x)$ and not $2\log x$.
- $\dfrac{\log_a M}{\log_a N} \neq \log_a \dfrac{M}{N}$. The left side is $\log_N M$ (by change of base!), a completely different object.
- $\log_a x^2 = 2\log_a |x|$, **not** $2\log_a x$, when x may be negative. In general $\log_a x^{2k} = 2k \log_a |x|$. Ignoring this loses solutions in equations and is the single most common error in this chapter.

Tricks that repay memorising

- **Golden ratio of logs:** $\log_a b = \dfrac{\ln b}{\ln a}$ — convert everything to ln (or to any one base) when a problem mixes several bases. Nine times out of ten the problem collapses.
- $\log_{1/a} b = \log_a \frac{1}{b} = -\log_a b$, and $\log_{1/a} \frac{1}{b} = \log_a b$.
- $\dfrac{1}{\log_a x} = \log_x a$: use this to merge sums like $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} = \log_N 2 + \log_N 3 = \log_N 6$.
- If $\log_a x = \log_b x$ $(x \neq 1)$ then $a = b$; if $\log_a x = \log_a y$ then $x = y$ (logs are one–one in both slots).

Solved Example 1 (classic Mains warm-up)

If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, find $\log_{10} 5$, $\log_{10} 6$, $\log_{10} \sqrt{72}$ and $\log_{10} 0.015$.

$\log 5 = \log \frac{10}{2} = 1 - 0.3010 = 0.6990$. $\log 6 = \log 2 + \log 3 = 0.7781$.
$\log \sqrt{72} = \frac{1}{2}\log(8 \cdot 9) = \frac{1}{2}(3\log 2 + 2\log 3) = \frac{1}{2}(0.9030 + 0.9542) = 0.9286$.
$\log 0.015 = \log \frac{15}{1000} = \log 3 + \log 5 - 3 = 0.4771 + 0.6990 - 3 = -1.8239$.

Solved Example 2 (merging reciprocals)

Evaluate $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$.

$= \log_{36} 2 + \log_{36} 3 = \log_{36} 6 = \log_{6^2} 6 = \frac{1}{2}$.

Solved Example 3 (chain rule of logs)

Evaluate $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{127} 128$.

The chain (L7) telescopes: the product equals $\log_2 128 = \log_2 2^7 = 7$.

Solved Example 4 (the star identity in action)

Simplify $7^{\log_7 5} - 5^{\log_7 7} + 3^{\log_9 16}$.

$7^{\log_7 5} = 5$ and $5^{\log_7 7} = 5^1 = 5$, so the first two terms cancel. For the third, $\log_9 16 = \log_{3^2} 4^2 = \log_3 4$, hence $3^{\log_9 16} = 3^{\log_3 4} = 4$. Answer: $\boxed{4}$.

Solved Example 5 (JEE Advanced 2011 – system with logs in exponents)

Let (x_0, y_0) be the solution of $(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$. Find x_0.

Take ln of the second equation: $\ln x \ln 3 = \ln y \ln 2$, so $\ln y = \dfrac{\ln 3}{\ln 2}\ln x$.
Take ln of the first: $\ln 2(\ln 2 + \ln x) = \ln 3(\ln 3 + \ln y) = \ln 3 \ln 3 + \dfrac{\ln^2 3}{\ln 2}\ln x$.
Collect $\ln x$: $\ln x \left(\ln 2 - \dfrac{\ln^2 3}{\ln 2}\right) = \ln^2 3 - \ln^2 2$, i.e. $\ln x \cdot \dfrac{\ln^2 2 - \ln^2 3}{\ln 2} = -(\ln^2 2 - \ln^2 3)$.
Hence $\ln x = -\ln 2$, giving $x_0 = \boxed{\frac{1}{2}}$.

4. Common Logarithms and Log Tables

Before calculators, *every* scientific multiplication, division, power and root was done with base-10 logarithms. JEE still expects you to use given values of $\log 2, \log 3, \log 7$ etc., and Olympiads love digit-counting problems — both rest on this section.

4.1 Characteristic and Mantissa

Write any positive number in **scientific form** $N = m \times 10^{c}$ with $1 \le m < 10$ and $c \in \mathbb{Z}$. Then

$$\log_{10} N = c + \log_{10} m, \qquad 0 \le \log_{10} m < 1.$$

Characteristic & mantissa

$$\log_{10} N = \underbrace{c}_{\text{characteristic (integer)}} + \underbrace{\log_{10} m}_{\text{mantissa}\in[0,1)}$$

- **Characteristic** of $N > 1$: (number of digits before the decimal point) $-\,1$.
- **Characteristic** of $0 < N < 1$: $-$(number of zeros immediately after the decimal point) $-\,1$.
- **Mantissa** depends only on the digit-sequence of N, never on the position of the decimal point: $\log 2.547$, $\log 25.47$, $\log 2547$ all share mantissa 0.4060.

Notation: $\log 0.02547 = \bar{2}.4060$ means $-2 + 0.4060$ ("bar two point four zero six zero"). The mantissa is always kept non-negative so that tables can be used.

Common Mistake

$\bar{2}.4060 \neq -2.4060$. In fact $\bar{2}.4060 = -2 + 0.4060 = -1.5940$. When you add or subtract barred logs, handle the integer and fractional parts separately. E.g. $\bar{1}.7 \div 2$: write $\bar{1}.7 = -1 + 0.7 = (-2) + 1.7$, then divide: $\bar{1}.85$.

4.2 How to Read a Log Table

A four-figure table gives mantissas of $\log_{10}$ for numbers 1.000 to 9.999. A row looks like this (row "25"):

	0	1	2	3	4	5	6	7	8	9	Mean differences	1	2	3	4	5	6	7	8	9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133		2	3	5	7	9	10	12	14	15

To find $\log 2547$**:**

Step 1. Scientific form: $2547 = 2.547 \times 10^{3}$, so characteristic $= 3$.

Step 2. Row **25**, column **4** (third significant digit): read 4048.

Step 3. Mean-difference column **7** (fourth digit): read 12. Add: $4048 + 12 = 4060$.

Step 4. Mantissa $= 0.4060$. Hence $\log 2547 = 3.4060$; likewise $\log 0.2547 = \bar{1}.4060$.

Antilog tables reverse the process: given mantissa 0.4060, the antilog table returns digits 2547; the characteristic then places the decimal point. $\text{antilog}(1.4060) = 25.47$, $\text{antilog}(\bar{2}.4060) = 0.02547$.

4.3 How Logs Crush Long Calculations

The whole point: logs convert $\times \to +$, $\div \to -$, powers $\to \times$, roots $\to \div$.

Solved Example 6 (a product–quotient by tables)

Evaluate $N = \dfrac{72.3 \times 0.0456}{8.91}$ using log tables.

$\log N = \log 72.3 + \log 0.0456 - \log 8.91$
$= 1.8591 + \bar{2}.6590 - 0.9499$
$= (1 - 2 - 0) + (0.8591 + 0.6590 - 0.9499) = -1 + 0.5682 = \bar{1}.5682.$
Antilog of .5682 gives digits 3700; characteristic $\bar{1}$ places the point: $N \approx \boxed{0.3700}$.
(Exact value 0.37003… — four-figure tables are accurate to about 4 significant figures.)

Solved Example 7 (a root that would be brutal by hand)

Find $(0.00437)^{1/5}$.

$\log(0.00437) = \bar{3}.6405 = -3 + 0.6405$. Divide by 5 — but first make the negative part a multiple of 5:

$$\bar{3}.6405 = -5 + 2.6405 \quad \Rightarrow \quad \tfrac{1}{5}(\bar{3}.6405) = -1 + 0.5281 = \bar{1}.5281.$$

Antilog of .5281 gives 3374; characteristic $\bar{1}$: answer $\approx \boxed{0.3374}$.

Solved Example 8 (compound growth / big powers)

Which is larger, 2^{100} or 10^{30}? How many digits does 2^{100} have? ($\log 2 = 0.3010$)

$\log 2^{100} = 100 \times 0.3010 = 30.10 > 30 = \log 10^{30}$, so $2^{100} > 10^{30}$.
Number of digits $= \lfloor \log_{10} 2^{100} \rfloor + 1 = \lfloor 30.10 \rfloor + 1 = \boxed{31}$.

Digit counting – an exam favourite

For a positive integer N: number of digits of $N = \lfloor \log_{10} N \rfloor + 1$.
For $0 < N < 1$: number of zeros between the decimal point and the first non-zero digit $= \lfloor -\log_{10} N \rfloor$ when $\log_{10} N \notin \mathbb{Z}$, i.e. $=$ |characteristic| $- 1$.

Solved Example 9 (zeros after the decimal point)

How many zeros come immediately after the decimal point in 3^{-50}? ($\log 3 = 0.4771$)

$\log 3^{-50} = -23.855 = \bar{24}.145$. Characteristic $= -24$, so there are $24 - 1 = \boxed{23}$ zeros before the first significant digit.

4.4 Why Different Bases Exist (and how to hop between them)

Base 10 matches our decimal writing system (hence tables and digit counts); base e matches calculus (hence limits/derivatives); base 2 matches doubling processes and computer science. Change of base (L5) means *no base is mathematically special* — all log functions are constant multiples of one another:

$$\log_2 x = \frac{\ln x}{\ln 2} \approx 1.4427 \ln x, \qquad \log_{10} x = \frac{\ln x}{\ln 10} \approx 0.4343 \ln x, \qquad \ln x \approx 2.3026 \log_{10} x.$$

Graphically: every $y = \log_a x$ curve is a vertical stretching (or flipping, if $0 < a < 1$) of every other one — they all pass through $(1, 0)$ and share the asymptote $x = 0$.

5. Domain and Range Problems

For $\log_{g(x)} f(x)$ to be defined we need **three simultaneous conditions**:

$$f(x) > 0, \qquad g(x) > 0, \qquad g(x) \neq 1.$$

Add to these any other structural demands (denominators $\neq 0$, even roots need ≥ 0, etc.), and *intersect* all the resulting sets.

Solved Example 10 (square root over a log)

Find the domain of $f(x) = \sqrt{\log_{0.3}(x-1)}$.

Need $x - 1 > 0$ and $\log_{0.3}(x-1) \geq 0$. Base $0.3 \in (0,1)$, so $\log_{0.3} t \geq 0 \iff 0 < t \leq 1$.
Thus $0 < x - 1 \leq 1$, i.e. $x \in \boxed{(1,\ 2]}$.

Solved Example 11 (variable base)

Find the domain of $f(x) = \log_{(x-2)}(x^2 - 6x + 8)$.

Conditions: (i) $x^2-6x+8 > 0 \Rightarrow (x-2)(x-4) > 0 \Rightarrow x < 2$ or $x > 4$; (ii) base $x-2 > 0 \Rightarrow x > 2$; (iii) $x - 2 \neq 1 \Rightarrow x \neq 3$.
Intersecting: $x > 4$. Domain $= \boxed{(4, \infty)}$ (condition (iii) is automatically satisfied there).

Solved Example 12 (nested domain)

Find the domain of $f(x) = \log_2\big(\log_3(\log_4 x)\big)$.

Peel from outside in: need $\log_3(\log_4 x) > 0 \Rightarrow \log_4 x > 1 \Rightarrow x > 4$. (The inner conditions $x > 0$, $\log_4 x > 0$ are weaker.) Domain $= \boxed{(4, \infty)}$.

Solved Example 13 (range via completing the square)

Find the range of $f(x) = \log_2(x^2 + 4x + 7)$.

$x^2 + 4x + 7 = (x+2)^2 + 3 \in [3, \infty)$. Since $\log_2$ is increasing and continuous, the range is $[\log_2 3,\ \infty)$.

Solved Example 14 (range of a rational-argument log)

Find the range of $f(x) = \log_e \dfrac{x^2 + e}{x^2 + 1}$.

Let $t = \dfrac{x^2 + e}{x^2 + 1} = 1 + \dfrac{e-1}{x^2+1}$. As x^2 runs over $[0, \infty)$, $\dfrac{e-1}{x^2+1} \in (0, e-1]$, so $t \in (1, e]$.
Hence $f = \ln t \in (0, 1]$. Range $= \boxed{(0,\ 1]}$.

Domain workflow

Write the three log conditions *first*, before touching the algebra. For $0 <$ base < 1 remember all inequality directions flip when you exponentiate. For nested logs, work **outermost condition first** — it usually dominates all inner ones.

6. Logarithmic Equations

Master strategies

1. **Same base, equate arguments:** $\log_a f = \log_a g \iff f = g > 0$.
2. **Definition form:** $\log_a f = k \iff f = a^k$ (with $f > 0$ automatic if a^k makes sense).
3. **Substitution:** if the equation contains $\log x$ polynomially, put $t = \log x$.
4. **One base to rule them all:** convert mixed bases with $\log_a M = \frac{\ln M}{\ln a}$.
5. **Take logs of both sides:** for equations with x in exponents, e.g. $x^{\log x} = 100x$.
6. **Check every candidate** against the original domain. Squaring, using $\log x^2 = 2\log x$, or multiplying by expressions that may vanish can create fake roots.

Solved Example 15 (substitution)

Solve $(\log_{10} x)^2 - 3\log_{10} x + 2 = 0$.

Let $t = \log_{10} x$: $t^2 - 3t + 2 = 0 \Rightarrow t = 1, 2 \Rightarrow x = 10,\ 100$. Both are > 0: $\boxed{x \in \{10, 100\}}$.

Solved Example 16 (x in the exponent)

Solve $x^{\log_{10} x} = 100x$, $x > 0$.

Take $\log_{10}$: $(\log x)^2 = 2 + \log x$. With $t = \log x$: $t^2 - t - 2 = 0 \Rightarrow t = 2, -1$.
$x = 100$ or $x = \frac{1}{10}$. Both valid: $\boxed{x \in \{100, \frac{1}{10}\}}$.

Solved Example 17 (mixed bases)

Solve $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{4}$.

All bases are powers of 2: $\log_4 x = \frac{1}{2}\log_2 x$, $\log_{16} x = \frac{1}{4}\log_2 x$.
$\log_2 x\left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{21}{4} \Rightarrow \log_2 x \cdot \frac{7}{4} = \frac{21}{4} \Rightarrow \log_2 x = 3 \Rightarrow \boxed{x = 8}$.

Solved Example 18 (extraneous root trap)

Solve $\log_{10}(x - 3) + \log_{10}(x + 6) = 1$.

Combine: $(x - 3)(x + 6) = 10 \Rightarrow x^2 + 3x - 28 = 0 \Rightarrow x = 4, -7$.
Domain demands $x > 3$. Reject -7. $\boxed{x = 4}$.

Solved Example 19 (symmetric exponential-log equation)

Solve $5^{\log_{10} x} + x^{\log_{10} 5} = 50$.

By the star identity (L9), $x^{\log_{10} 5} = 5^{\log_{10} x}$. So $2 \cdot 5^{\log_{10} x} = 50 \Rightarrow 5^{\log_{10} x} = 25 \Rightarrow \log_{10} x = 2 \Rightarrow \boxed{x = 100}$.

Solved Example 20 (JEE Advanced 2012 – the famous nested radical)

Find the value of

$$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \cdots}}}\right).$$

Let $y = \sqrt{4 - \frac{1}{3\sqrt{2}}\,y}$ (self-similarity of the infinite radical), $y > 0$. Then $y^2 + \frac{y}{3\sqrt{2}} - 4 = 0 \Rightarrow 3\sqrt{2}\,y^2 + y - 12\sqrt{2} = 0$.
Discriminant $= 1 + 4 \cdot 3\sqrt{2} \cdot 12\sqrt{2} = 1 + 288 = 289 = 17^2$, so $y = \dfrac{-1+17}{6\sqrt{2}} = \dfrac{16}{6\sqrt{2}} = \dfrac{8}{3\sqrt{2}}$.
The bracket is $\dfrac{y}{3\sqrt{2}} = \dfrac{8}{18} = \dfrac{4}{9}$. Now $\log_{3/2}\dfrac{4}{9} = \log_{3/2}\left(\dfrac{3}{2}\right)^{-2} = -2$.
Answer: $6 + (-2) = \boxed{4}$.

7. Nested Logarithms

"Nested" means logs inside logs: $\log_a \log_b \log_c(\cdot)$. Two rules organise everything:

Unwrapping nested logs

- **Evaluate inside-out:** compute the innermost log first.
- **Solve outside-in:** to solve $\log_a(\text{stuff}) = k$, rewrite as $\text{stuff} = a^k$ and repeat.
- **Domain cascades:** $\log_b u$ needs $u > 0$; $\log_a \log_b u$ needs $\log_b u > 0$, i.e. $u > 1$ (if $b > 1$); three layers need $u > b$; and so on. Each extra layer strengthens the condition.

Solved Example 21 (inside-out evaluation)

Evaluate $\log_2 \log_2 \log_2 65536$.

$65536 = 2^{16}$, so innermost $= 16$. Next: $\log_2 16 = 4$. Outermost: $\log_2 4 = \boxed{2}$.

Solved Example 22 (outside-in solving)

Solve $\log_2\big(\log_3(\log_4 x)\big) = 0$.

Outer layer: $\log_3(\log_4 x) = 2^0 = 1$. Next: $\log_4 x = 3$. Finally $x = 4^3 = \boxed{64}$.

Solved Example 23 (Olympiad-style nested system)

If $\log_2(\log_3 a) = \log_3(\log_2 b) = 1$, find a and compare a with b.

$\log_3 a = 2 \Rightarrow a = 9$. $\log_2 b = 3 \Rightarrow b = 8$. So $a = 9 > b = 8$ — despite the symmetric look, the answers are not symmetric. Order of composition matters.

Solved Example 24 (nested log identity chase)

If $a, b, c > 1$ satisfy $\log_a b = c$ and $\log_b c = a$, prove that $\log_c a = \dfrac{1}{ac}$.

By the chain rule (L7): $\log_a b \cdot \log_b c \cdot \log_c a = \log_a a = 1$. Substituting: $c \cdot a \cdot \log_c a = 1$, hence $\log_c a = \frac{1}{ac}$. ■

8. Logarithmic Inequalities

This is where most marks are lost. Everything hinges on monotonicity:

The direction rules

Base $a > 1$: $\log_a f > \log_a g \iff f > g > 0$ **Base** $0 < a < 1$: $\log_a f > \log_a g \iff 0 < f < g$

$a > 1$: $\log_a f > k \iff f > a^k$ $0 < a < 1$: $\log_a f > k \iff 0 < f < a^k$

Golden rule: removing a log with base in $(0, 1)$ flips the inequality; base > 1 preserves it. Domain conditions ($f > 0$, $g > 0$) must be carried along at every step.

Solved Example 25 (base less than 1)

Solve $\log_{0.5}(x^2 - 5x + 6) > -1$.

Base $\in (0, 1)$: the inequality unwraps to $0 < x^2 - 5x + 6 < (0.5)^{-1} = 2$.
Left part: $(x-2)(x-3) > 0 \Rightarrow x < 2$ or $x > 3$. Right part: $x^2 - 5x + 4 < 0 \Rightarrow (x-1)(x-4) < 0 \Rightarrow 1 < x < 4$.
Intersection: $\boxed{x \in (1, 2) \cup (3, 4)}$.

Solved Example 26 (variable base — the two-case split)

Solve $\log_x(2x) \geq 2$ for permissible x.

Domain: $x > 0$, $x \neq 1$ (then $2x > 0$ automatically).
Case 1: $x > 1$. Then $2x \geq x^2 \Rightarrow x(x-2) \leq 0 \Rightarrow 0 \leq x \leq 2$. With $x > 1$: $1 < x \leq 2$.
Case 2: $0 < x < 1$. Inequality flips: $2x \leq x^2 \Rightarrow x \geq 2$ or $x \leq 0$ — impossible here.
Answer: $\boxed{x \in (1, 2]}$.

Solved Example 27 (exponential-with-log inequality)

Solve $x^{\log_2 x} > 16$, $x > 0$.

Take $\log_2$ (base $2 > 1$, direction preserved): $(\log_2 x)^2 > 4 \Rightarrow \log_2 x > 2$ or $\log_2 x < -2$.
$\boxed{x \in \left(0, \frac{1}{4}\right) \cup (4, \infty)}$.

Inequality checklist

1) Write the domain first. 2) Identify the base: > 1 keep direction, $\in (0, 1)$ flip, variable base $\Rightarrow$ split cases. 3) Solve each polynomial inequality by the wavy-curve/sign-scheme method. 4) Intersect with the domain at the very end. Never cancel logs across an inequality without checking the base.

Common Mistake

From $\log_a f > \log_a g$ students write $f > g$ and forget $g > 0$. Example: $\log_2(x) > \log_2(x^2 - 2)$ needs $x > x^2 - 2$ **and** $x^2 - 2 > 0$, giving $(\sqrt{2}, 2)$, not $(-1, 2)$.

9. Limits Involving Logarithmic and Exponential Functions

9.1 The Standard Limits

Standard limits — the complete JEE toolkit

$$\text{(S1)}\quad \lim_{x\to 0}\frac{e^x - 1}{x} = 1 \qquad \text{(S2)}\quad \lim_{x\to 0}\frac{a^x - 1}{x} = \ln a \quad (a > 0)$$

$$\text{(S3)}\quad \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1 \qquad \text{(S4)}\quad \lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e = \frac{1}{\ln a}$$

$$\text{(S5)}\quad \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e \qquad \text{(S6)}\quad \lim_{x\to 0}(1+x)^{1/x} = e$$

$$\text{(S7)}\quad \lim_{x\to 0}\frac{(1+x)^p - 1}{x} = p \qquad \text{(S8)}\quad \lim_{x\to\infty}\frac{\ln x}{x^p} = 0,\ \lim_{x\to 0^+} x^p \ln x = 0 \quad (p > 0)$$

1^∞ **form:** if $f \to 1$ and $g \to \infty$, then $\lim f^{\,g} = e^{\lim g\,(f-1)}$ (whenever the latter limit exists).

(S8) is the **hierarchy of growth**: as $x \to \infty$,

$$\ln x \ \ll\ x^p \text{ (any } p > 0) \ \ll\ a^x \text{ (any } a > 1) \ \ll\ x! \ \ll\ x^x.$$

Logs grow slower than every power; exponentials beat every power. This single line settles many limit questions instantly.

Solved Example 28 (S2 and S3 together)

Evaluate $\displaystyle\lim_{x\to 0}\frac{5^x - 3^x}{\ln(1+2x)}$.

$$\lim_{x\to 0}\frac{(5^x-1)-(3^x-1)}{x}\cdot\frac{x}{\ln(1+2x)}=(\ln 5-\ln 3)\cdot\frac{1}{2}=\boxed{\tfrac{1}{2}\ln\tfrac{5}{3}}.$$

Solved Example 29 (1^∞ form)

Evaluate $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^x$.

$f=\frac{x+3}{x+1}\to 1$, $g=x\to\infty$. Then $g(f-1)=x\cdot\frac{2}{x+1}\to 2$, so the limit is $\boxed{e^2}$.

Solved Example 30 (log-power race)

Evaluate $\lim_{x\to 0^+}x\ln x$ and $\lim_{x\to 0^+}x^x$.

Put $x=e^{-t}$, $t\to\infty$: $x\ln x=-te^{-t}\to 0$ (exponential beats polynomial). Hence $x^x=e^{x\ln x}\to e^0=\boxed{1}$. The graph of $y=x^x$ approaches height 1 at the origin with a minimum at $x=1/e$.

Solved Example 31 (JEE-style combined limit)

Evaluate $\lim_{x\to 0}\frac{e^{\sin x}-e^x}{\sin x-x}$.

Write $e^{\sin x}-e^x=e^x\big(e^{\sin x-x}-1\big)$. With $u=\sin x-x\to 0$,

$$\lim_{x\to 0}e^x\cdot\frac{e^u-1}{u}=1\cdot 1=\boxed{1}.$$

No series expansion needed — factoring plus (S1) does it.

9.2 Continuity

e^x, a^x are continuous on all of $\mathbb{R}$; $\ln x$, $\log_a x$ are continuous on $(0,\infty)$. Compositions like $\ln(f(x))$ are continuous wherever f is continuous *and* positive. Typical exam task: choose a constant to repair a removable discontinuity.

Solved Example 32 (repairing continuity at a point)

$f(x)=\dfrac{\ln(1+ax)-\ln(1-bx)}{x}$ for $x\neq 0$. Find $f(0)$ so that f is continuous at 0.

$\lim_{x\to 0}\left[a\cdot\frac{\ln(1+ax)}{ax}+b\cdot\frac{\ln(1-bx)}{-bx}\right]=a+b$. So define $f(0)=\boxed{a+b}$.

Solved Example 33 (continuity with a parameter in an exponent)

For what value of k is $f(x)=\begin{cases}(1+3x)^{1/x}, & x\neq 0\\ k, & x=0\end{cases}$ continuous at 0?

$(1+3x)^{1/x}=\big[(1+3x)^{1/(3x)}\big]^3\to e^3$. Hence $k=\boxed{e^3}$.

9.3 Differentiability and Derivatives

Derivatives

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}a^x = a^x \ln a, \qquad \frac{d}{dx}\ln x = \frac{1}{x}\ (x>0), \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}\quad (x\neq 0), \qquad \frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}.$$

Logarithmic differentiation: for $y = f(x)^{g(x)}$, write $\ln y = g\ln f$ and differentiate:

$$\frac{y'}{y} = g'\ln f + \frac{gf'}{f}.$$

$\ln x$ and e^x are differentiable everywhere on their domains — non-differentiability in this chapter enters only through **modulus** and **piecewise** constructions.

Solved Example 34 (the classic x^x)

Differentiate $y = x^x$, $x>0$, and find its minimum.

$\ln y = x\ln x \Rightarrow \dfrac{y'}{y} = \ln x + 1 \Rightarrow y' = x^x(1+\ln x)$.
$y' = 0 \iff \ln x = -1 \iff x = e^{-1}$; sign of y' changes $-\to+$ there, so minimum value $= (1/e)^{1/e} \approx 0.6922$ at $x = \boxed{1/e}$.

Solved Example 35 (points of non-differentiability)

Find all points where $f(x) = \big|\ln|x|\big|$ is not differentiable.

Domain: $x\neq 0$. $\ln|x|$ is smooth on its domain; the outer modulus creates corners exactly where $\ln|x| = 0$, i.e. $|x| = 1$. So f fails to be differentiable at $x = \boxed{\pm 1}$ (and $x=0$ is outside the domain, so it is not counted).

Solved Example 36 (derivative of a general exponential tower)

If $y = (\sin x)^{\ln x}$ on an interval where $\sin x > 0$, find y'.

$\ln y = \ln x \cdot \ln(\sin x)$. Differentiate: $\dfrac{y'}{y} = \dfrac{\ln(\sin x)}{x} + \ln x\cot x$, so $y' = (\sin x)^{\ln x}\left[\dfrac{\ln(\sin x)}{x} + \ln x\cot x\right]$.

When to reach for logarithmic differentiation

Use it whenever (i) the variable sits in both base and exponent, (ii) the function is a big product/quotient of powers — ln converts it to a sum you can differentiate term-by-term, or (iii) you must compare growth rates (compare ln of both quantities instead). Example: which is bigger, e^π or π^e? Compare $\frac{\ln x}{x}$ at $x = e, \pi$: since $\frac{\ln x}{x}$ is decreasing for $x > e$, $\frac{\ln e}{e} > \frac{\ln\pi}{\pi} \Rightarrow \pi\ln e > e\ln\pi \Rightarrow e^\pi > \pi^e$.

10. Olympiad Corner

Solved Example 37 (telescoping with reciprocals)

Prove that $\dfrac{1}{\log_2 100!}+\dfrac{1}{\log_3 100!}+\cdots+\dfrac{1}{\log_{100} 100!}=1$.

Each term $\dfrac{1}{\log_k 100!}=\log_{100!} k$. The sum is $\log_{100!}(2\cdot 3\cdots 100)=\log_{100!} 100!=1$. ■

Solved Example 38 (AMC-flavoured)

If $\log_2 x+\log_4 x+\log_8 x=11$, find x.

$\log_2 x\left(1+\frac{1}{2}+\frac{1}{3}\right)=11\Rightarrow \log_2 x\cdot\frac{11}{6}=11\Rightarrow \log_2 x=6\Rightarrow x=\boxed{64}$.

Solved Example 39 (irrationality via logarithms)

Prove that $\log_{10} 2$ is irrational.

Suppose $\log_{10} 2=\frac{p}{q}$ with $p,q\in\mathbb{Z}^+$. Then $10^p=2^q$, i.e. $2^p5^p=2^q$. Unique factorisation forces $p=q$ and $p=0$ — contradiction. Hence $\log_{10} 2$ is irrational. ■
(The same argument shows $\log_m n$ is irrational whenever $m,n\geq 2$ have different prime supports.)

Solved Example 40 (a bounding argument)

Without tables, prove $\frac{3}{10}<\log_{10} 2<\frac{1}{3}$.

$2^{10}=1024>10^3\Rightarrow 10\log 2>3\Rightarrow \log 2>0.3$.
$2^3=8<10\Rightarrow 3\log 2<1\Rightarrow \log 2<\frac{1}{3}$. ■
Olympiad problems often ask for such rational bounds; powers of 2 near powers of 10 ($2^{10}\approx 10^3$, $2^{93}\approx 10^{28}$) are the levers.

Solved Example 41 (functional-equation flavour)

Find all continuous $f:(0,\infty)\to\mathbb{R}$ with $f(xy)=f(x)+f(y)$ for all $x,y>0$.

Define $g(t)=f(e^t)$. Then $g(s+t)=g(s)+g(t)$ (Cauchy's equation); continuity forces $g(t)=ct$. Hence $f(x)=c\ln x$. *The logarithm is characterised by its product-to-sum law* — this is the deep reason logs appear wherever multiplicative structure must be linearised (entropy, pH, decibels, complexity theory). ■

11. Practice Exercises

Use $\log_{10} 2=0.3010$, $\log_{10} 3=0.4771$, $\log_{10} 7=0.8451$ where needed.

Exercise A — JEE Mains Level

A1. Evaluate $\log_{\sqrt{8}} 128$.

A2. If $\log_{10} x + \log_{10} x^2 + \log_{10} x^3 = 6$, find x.

A3. Simplify $\dfrac{1}{\log_3 60} + \dfrac{1}{\log_4 60} + \dfrac{1}{\log_5 60}$.

A4. Find the number of digits in 6^{20}.

A5. Compute $\log_{10} 0.072$ in terms of the given data.

A6. Solve $\log_2(x^2 - 1) = 3$.

A7. Find the domain of $f(x) = \log_{10} \dfrac{5x - x^2}{4}$.

A8. Find the domain of $f(x) = \dfrac{1}{\ln(2 - x)} + \sqrt{x + 1}$.

A9. Solve $4^x - 3 \cdot 2^{x+1} + 8 = 0$.

A10. Evaluate $\displaystyle\lim_{x \to 0} \frac{e^{3x} - 1}{\ln(1 + 6x)}$.

A11. Evaluate $\displaystyle\lim_{x \to \infty} \left(1 + \frac{3}{x}\right)^{2x}$.

A12. If $f(x) = \dfrac{a^x - 1}{x}$ for $x \neq 0$ and f is continuous at 0 with $f(0) = \ln 9$, find a.

A13. Differentiate $y = \log_{10}(\sin x)$ on an interval where $\sin x > 0$.

A14. Find $\dfrac{dy}{dx}$ for $y = x^{\sqrt{x}}$, $x > 0$.

A15. Arrange in increasing order: $\log_2 3$, $\log_4 15$, $\log_4 18$.

Exercise B — JEE Advanced Level

B1. Solve $\log_2 x + \log_x 2 = \dfrac{10}{3}$.

B2. Solve the inequality $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$.

B3. If $\log_{12} 27 = a$, express $\log_6 16$ in terms of a.

B4. Solve $\big(\log_2 x\big)^2 + \log_2 x^2 - 3 \le 0$ (note: $\log_2 x^2$, not $(\log_2 x)^2$).

B5. Find all x with $x^{(\log_{10} x)^2 - 3\log_{10} x + 1} > 1000$.

B6. Solve $\log_3\big(\log_2(x^2 - 4x + 5)\big) \le 0$; give the solution set.

B7. Evaluate $\displaystyle\lim_{x \to 0^+} (\sin x)^{\tan x}$.

B8. Find all points of non-differentiability of $f(x) = \big|\, e^{|x|} - 2 \,\big|$.

B9. The number $N = \dfrac{\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{99} 100}{\log_2 100}$ — evaluate N.

B10. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, prove that $\dfrac{1}{1+x} + \dfrac{1}{1+y} + \dfrac{1}{1+z} = 1$.

Exercise C — Olympiad Level

C1. Prove that $\log_2 3 > \log_3 5$.

C2. Prove that for $n \geq 2$, $\dfrac{1}{\log_2 n!} + \dfrac{1}{\log_3 n!} + \cdots + \dfrac{1}{\log_n n!} = 1$.

C3. Show that $\log_{10} 5 \cdot \log_{10} 20 + (\log_{10} 2)^2 = 1$.

C4. (AMC 2019 adapted) Positive reals x, y satisfy $\log_2 x = \log_y 16$ and $xy = 64$. Find $\left(\log_2 \frac{x}{y}\right)^2$.

C5. Prove that $\log_2 3$ and $\log_3 2$ cannot both be expressed with any common rational: i.e. show $\log_2 3$ is irrational and deduce $\log_3 2$ is too.

C6. Find all real solutions of $3^x + 4^x = 5^x$, and prove there are no others (hint: divide by 5^x and use monotonicity of exponentials with base in $(0,1)$).

C7. If $a, b, c > 1$, prove that $\log_a b + \log_b c + \log_c a \;\geq\; 3$, with equality iff $a = b = c$.

C8. How many digits does 125^{100} have?

Answer Key

A1. 14/3. **A2.** $x = 10$. **A3.** $\log_{60}(3 \cdot 4 \cdot 5) = \log_{60} 60 = 1$. **A4.** $20 \log 6 = 20(0.7781) = 15.562 \Rightarrow$ 16 digits. **A5.** $\log 72 - 3 = 3\log 2 + 2\log 3 - 3 = -1.1428$, i.e. $\bar{2}.8572$. **A6.** $x = \pm 3$. **A7.** need $\frac{5x - x^2}{4} > 0 \Rightarrow (0, 5)$. **A8.** $[-1, 2) \setminus \{1\}$. **A9.** $2^x = 2, 4 \Rightarrow x = 1, 2$. **A10.** 1/2. **A11.** e^6. **A12.** $\ln a = \ln 9 \Rightarrow a = 9$. **A13.** $\cot x \cdot \log_{10} e$. **A14.** $x^{\sqrt{x}} \cdot \frac{\ln x + 2}{2\sqrt{x}}$. **A15.** $\log_2 3 < \log_4 15 < \log_4 18$ (write all with base 2: $\log_2 3 = \log_2 \sqrt{9}$ and $\sqrt{9} < \sqrt{15} < \sqrt{18}$).

B1. $t + \frac{1}{t} = \frac{10}{3} \Rightarrow t = 3, \frac{1}{3} \Rightarrow x = 8,\ 2^{1/3}$. **B2.** $\log_{0.09} = \frac{1}{2}\log_{0.3}$; with $t = \log_{0.3}(x-1)$: $t < \frac{t}{2} \Rightarrow t < 0 \Rightarrow x - 1 > 1 \Rightarrow x > 2$. **B3.** $a = \frac{3\log_2 3}{2 + \log_2 3} \Rightarrow \log_2 3 = \frac{2a}{3-a}$; then $\log_6 16 = \frac{4}{1 + \log_2 3} = \frac{4(3-a)}{3+a}$. **B4.** $t^2 + 2t - 3 \leq 0 \Rightarrow -3 \leq t \leq 1 \Rightarrow x \in [\frac{1}{8}, 2]$. **B5.** take $\log_{10}$: with $t = \log x$, $t(t^2 - 3t + 1) > 3 \Rightarrow t^3 - 3t^2 + t - 3 > 0 \Rightarrow (t-3)(t^2+1) > 0 \Rightarrow t > 3 \Rightarrow x > 1000$. **B6.** $0 < \log_2(x^2 - 4x + 5) \leq 1 \Rightarrow 1 < x^2 - 4x + 5 \leq 2 \Rightarrow 0 < (x-2)^2 \leq 1 \Rightarrow x \in [1, 3] \setminus \{2\}$. **B7.** $e^{\lim \tan x \ln \sin x} = e^0 = 1$ since $\tan x \ln \sin x \sim x \ln x \to 0$. **B8.** inner corner of $e^{|x|}$ at $x = 0$ survives (there $e^{|x|} - 2 < 0$); outer modulus adds corners where $e^{|x|} = 2$: non-differentiable at $x = 0, \pm \ln 2$. **B9.** numerator telescopes to $\log_2 100$, so $N = 1$. **B10.** $\frac{1}{1+x} = \log_{abc} a$ etc.; sum $= \log_{abc} abc = 1$.

C1. $\log_2 3 > \frac{3}{2}$ since $3^2 = 9 > 8 = 2^3$; and $\log_3 5 < \frac{3}{2}$ since $5^2 = 25 < 27 = 3^3$. Hence $\log_2 3 > \frac{3}{2} > \log_3 5$. **C2.** each term is $\log_{n!} k$; sum $= \log_{n!}(2 \cdot 3 \cdots n) = \log_{n!} n! = 1$. **C3.** $\log 5 \log 20 + \log^2 2 = (1 - \log 2)(1 + \log 2) + \log^2 2 = 1 - \log^2 2 + \log^2 2 = 1$. **C4.** $u = \log_2 x$, $v = \log_2 y$: $u = \frac{4}{v}$ and $u + v = 6$, so $uv = 4$ and $\left(\log_2 \frac{x}{y}\right)^2 = (u - v)^2 = 36 - 16 = \boxed{20}$. **C5.** $\log_2 3 = \frac{p}{q} \Rightarrow 2^p = 3^q$: LHS even, RHS odd — contradiction. Then $\log_3 2 = 1/\log_2 3$ is also irrational (reciprocal of an irrational is irrational). **C6.** $x = 2$; divide by 5^x: $g(x) = (3/5)^x + (4/5)^x$ is strictly decreasing (sum of decreasing exponentials), so $g(x) = 1$ at most once; $g(2) = \frac{9}{25} + \frac{16}{25} = 1$. **C7.** the three terms are positive (all bases and arguments exceed 1) and their product is $\log_a b \cdot \log_b c \cdot \log_c a = 1$ by the chain rule; AM–GM gives sum $\geq 3\sqrt[3]{1} = 3$, equality iff all terms equal 1, i.e. $a = b = c$. **C8.** $125^{100} = 5^{300}$; $300 \log 5 = 300(1 - 0.3010) = 209.7 \Rightarrow \lfloor 209.7 \rfloor + 1 = 210$ digits.

On the harder key entries

Full step-by-step solutions to Exercises B and C are ideal material for a follow-up "Solutions & Discussion" booklet or a live problem-solving session — several (B8, C6, C7) contain genuinely instructive subtleties about monotonicity and modulus corners.

12. One-Page Formula Sheet

Definitions & identities

$\log_a N = x \iff a^x = N \quad (a > 0, a \neq 1, N > 0) \quad \cdot \quad \log_a 1 = 0, \ \log_a a = 1, \ a^{\log_a N} = N, \ \log_a a^x = x$

$\log(MN) = \log M + \log N \quad \cdot \quad \log \frac{M}{N} = \log M - \log N \quad \cdot \quad \log_a M^p = p \log_a M \quad \cdot \quad \log_{a^q} M^p = \frac{p}{q} \log_a M$

$\log_a M = \dfrac{\log_b M}{\log_b a} \quad \cdot \quad \log_a b = \dfrac{1}{\log_b a} \quad \cdot \quad \log_a b \cdot \log_b c = \log_a c \quad \cdot \quad a^{\log_b c} = c^{\log_b a}$

$\log_a x^{2k} = 2k \log_a |x| \quad \cdot \quad$ digits of $N = \lfloor \log_{10} N \rfloor + 1$

Graphs, signs, inequalities

Domain of $\log_a x$: $(0, \infty)$; range $\mathbb{R}$; passes $(1, 0), (a, 1)$; asymptote $x = 0$.
$a > 1$: increasing; $0 < a < 1$: decreasing. $\log_a x > 0 \iff a, x$ same side of 1.
$a > 1$: $\log_a f > \log_a g \iff f > g > 0$; $0 < a < 1$: $\log_a f > \log_a g \iff 0 < f < g$.
Variable base: split into cases base > 1 and $0 <$ base < 1; state domain first.

Calculus

$\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1, \quad \lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \ln a, \quad \lim\limits_{x \to 0} \dfrac{\ln(1 + x)}{x} = 1, \quad \lim\limits_{x \to 0} (1 + x)^{1/x} = e, \quad \lim\limits_{x \to \infty} \dfrac{\ln x}{x^p} = 0 \ (p > 0)$

1^∞: $\lim f^g = e^{\lim g(f-1)}$. $\quad (e^x)' = e^x, \ (a^x)' = a^x \ln a, \ (\ln x)' = \frac{1}{x}, \ (\log_a x)' = \frac{1}{x \ln a}, \ (\ln |x|)' = \frac{1}{x}$.

$y = f^g \Rightarrow y' = f^g \left(g' \ln f + \frac{g f'}{f} \right)$. $\quad$ Growth: $\ln x \ll x^p \ll a^x$ $(p > 0, \ a > 1)$.

Log tables in 4 lines

$\log N =$ characteristic + mantissa, mantissa $\in [0, 1)$ from tables; characteristic from decimal position.
$N > 1$: characteristic = digits before point -1; $N < 1$: $= -($leading zeros $+ 1)$ (write with bar).
Multiply/divide $\to$ add/subtract logs; powers/roots $\to$ multiply/divide logs; antilog to finish.
Before dividing a barred log, rewrite so the negative part is divisible: $\bar{3}.6405 = \bar{5} + 2.6405$.

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