# Complex Numbers

The Complete Picture — Algebra, Trigonometry & Geometry, Unified

Complete Edition — All Eight Parts

SIMPLIFIED EDUCATION
simplifiededucation.in | sunilyadav.org

# Contents


## Part 1: Foundations — Where Algebra Meets Geometry 1

**1 Why Do We Need Complex Numbers? 1**
- 1.1 The Problem That Started It All . . . 1
- 1.2 The Definition . . . 1

**2 The Algebra of Complex Numbers 2**
- 2.1 Powers of $i$: A Cycle of Four . . . 2
- 2.2 Addition, Subtraction, Multiplication . . . 2
- 2.3 The Conjugate . . . 2
- 2.4 Division: Realizing the Denominator . . . 3

**3 The Argand Plane: Complex Numbers as Points 3**
- 3.1 Addition as Vector Addition . . . 4

**4 Modulus: Distance in the Complex Plane 4**

**5 Argument: Direction in the Complex Plane 5**

**6 Geometric Meaning of the Conjugate 6**

**7 Worked Examples: Connecting the Threads 6**

## Part 2: Polar Form, De Moivre's Theorem & the Trigonometry Machine 8

**1 From Cartesian to Polar: The Natural Language of Rotation 8**

**2 Multiplication in Polar Form: The Key Theorem 8**
- 2.1 Division in Polar Form . . . 9

**3 Euler's Formula: A Change of Notation 9**

**4 De Moivre's Theorem 10**

**5 The Payoff: Generating Trigonometric Identities on Demand 10**
- 5.1 Multiple-Angle Formulas: $\cos n\theta$ and $\sin n\theta$ in Terms of $\cos\theta, \sin\theta$ . . . 10
- 5.2 Reversing the Process: Powers of $\sin\theta, \cos\theta$ in Terms of Multiple Angles . . . 11
- 5.3 $\tan n\theta$ in Terms of $\tan\theta$ . . . 11
- 5.4 Summing Trigonometric Series: The Geometric Series Trick . . . 12
- 5.5 Sum-to-Product and Product-to-Sum, Revisited . . . 12

**6 Worked Examples: Putting It All Together 12**

## Part 3: $n$th Roots of Unity — Algebra Made Geometry 14

**1 Solving $z^n = 1$ 14**

**2 The Geometry: A Regular Polygon for Free 14**

**3 Algebraic Structure 15**
- 3.1 Primitive Roots . . . 16

**4 $n$th Roots of a General Complex Number 16**

**5 Factoring $z^n - 1$ and Related Polynomials** **16**

**6 The Roots-of-Unity Filter** **17**

## Part 4: Trigonometric Equations & Periods — Solved Completely 19

**1 Why Complex Numbers Solve Trigonometric Equations Better** **19**

**2 The Three Classical General Solutions, Derived** **19**
2.1 $\cos\theta = \cos\alpha$ . . . 19
2.2 $\sin\theta = \sin\alpha$ . . . 19
2.3 $\tan\theta = \tan\alpha$ . . . 20

**3 Equations Involving Multiple Angles** **20**
3.1 $\sin n\theta = 0$ and $\cos n\theta = 0$ . . . 20
3.2 $\cos n\theta = \cos m\phi$-type and Compound Equations . . . 20

**4 Periods: Made Rigorous by Complex Exponentials** **21**
4.1 The Problem With Guessing Periods . . . 21
4.2 Rewriting Any Trigonometric Polynomial via Euler's Formula . . . 21
4.3 Periods of Non-Polynomial Combinations . . . 22

**5 Worked Examples: End to End** **22**

## Part 5: Geometry with Complex Numbers — Coordinate-Free Proof 24

**1 The Toolkit So Far, Repurposed for Geometry** **24**

**2 The Section Formula** **24**

**3 Rotation: The Signature Move** **25**

**4 Collinearity and the Equilateral Triangle Condition** **25**
4.1 When Are Three Points Collinear? . . . 25
4.2 The Equilateral Triangle Condition . . . 26

**5 Perpendicularity and Parallelism** **26**

**6 Concyclicity: Four Points on a Circle** **26**

**7 Area of a Triangle** **27**

**8 Loci in the Argand Plane** **27**

**9 Worked Examples: Full Problems** **28**

## Part 6: Coordinate Geometry & Conics in Complex Form 30

**1 The General Equation of a Line** **30**

**2 The General Equation of a Circle** **30**

**3 Tangents to a Circle** **31**

**4 Reflection of a Point in a Line** **32**

**5 Conics: Complex Parametrization** **32**

**6 Worked Examples** **33**

## Part 7: Equations with Complex Roots 35

**1 The Fundamental Theorem of Algebra** **35**

**2 Complex Square Roots: A Prerequisite** **35**

**3 Quadratic Equations with Complex Coefficients** **36**

**4 Real-Coefficient Polynomials: The Conjugate Root Theorem** **36**

**5 Vieta's Formulas, Extended to Complex Roots** **37**

**6 Equations Reducible by Substitution** **38**
6.1 Reciprocal Equations . . . . . 38

**7 Where Do the Roots Live? Bounding Root Modulus** **38**

**8 Worked Examples** **39**

## Part 8: Olympiad Techniques — The Complete Toolkit in Action 41

**1 Sharpening the Triangle Inequality** **41**

**2 The Complex Cauchy–Schwarz Inequality** **42**

**3 The Complex-Substitution Trick** **42**

**4 Extremal Modulus Problems** **42**

**5 Curated Problems: The Toolkit, Fully Assembled** **43**

**6 Closing Practice Set: Mixed, Cumulative** **44**

## Index 46

PART 1

# Foundations — Where Algebra Meets Geometry

## 1 Why Do We Need Complex Numbers?

### 1.1 The Problem That Started It All

Consider the equation

$$x^2 + 1 = 0 \quad \Longrightarrow \quad x^2 = -1.$$

No real number squares to $-1$, since the square of every real number is non-negative. For centuries this equation was simply declared to have "no solution." But something more interesting was brewing.

**A Surprising Historical Fact**

Complex numbers were *not* invented to solve $x^2 = -1$. They were forced into existence by **Cardano's formula** for cubic equations in the 1500s. Even when a cubic like $x^3 - 15x - 4 = 0$ had three genuine *real* roots (e.g. $x = 4$), applying Cardano's formula required taking the square root of a negative number *in the middle of the calculation*, even though the final answer was real. Mathematicians could not avoid $\sqrt{-1}$ even when they only cared about real answers. This is what made $i$ impossible to ignore.

### 1.2 The Definition

**Definition The Imaginary Unit**

We define a number $i$ with the property

$$i^2 = -1.$$

A **complex number** is any expression of the form

$$z = x + iy, \qquad x, y \in \mathbb{R},$$

where $x = \mathrm{Re}(z)$ is called the **real part** and $y = \mathrm{Im}(z)$ is called the **imaginary part**. The set of all such $z$ is denoted $\mathbb{C}$.

Real numbers are the special case $y = 0$; so $\mathbb{R} \subset \mathbb{C}$. When $x = 0$ and $y \neq 0$, $z = iy$ is called **purely imaginary**.

**Common Pitfall**

$i$ is *not* "an imaginary quantity that doesn't really exist." It is exactly as legitimate a mathematical object as $-1$ or $\sqrt{2}$ once was considered illegitimate. Every complex number is a perfectly well-defined point, as we will see in Section 3. The word "imaginary" is a historical accident, not a mathematical judgement.

# 2 The Algebra of Complex Numbers

## 2.1 Powers of $i$: A Cycle of Four

Since $i^2 = -1$, we get $i^3 = i^2 \cdot i = -i$, and $i^4 = i^2 \cdot i^2 = 1$. This creates a repeating cycle:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \ldots$$

**Reducing Any Power of $i$**

For any integer $n$, write $n = 4q + r$ where $r \in \{0, 1, 2, 3\}$ (i.e. $r = n \bmod 4$). Then

$$i^n = i^r.$$

This single fact will resurface constantly — it is the seed of the theory of *roots of unity* developed in Part 3.

**Example**

Find $i^{2025}$.
*Solution.* $2025 = 4(506) + 1$, so $i^{2025} = i^1 = i$.

## 2.2 Addition, Subtraction, Multiplication

Complex numbers are added and multiplied exactly like binomials in $x$, using $i^2 = -1$ to simplify. For $z_1 = a + ib$ and $z_2 = c + id$:

$$z_1 \pm z_2 = (a \pm c) + i(b \pm d)$$

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

**Remark**

Notice the multiplication rule is *not* "multiply real parts and multiply imaginary parts." The cross terms and the sign flip from $i^2 = -1$ are essential. Committing $(ac - bd) + i(ad + bc)$ to memory is useful, but re-deriving it by direct expansion takes ten seconds and avoids errors.

## 2.3 The Conjugate

**Definition Complex Conjugate**

The conjugate of $z = x + iy$ is

$$\bar{z} = x - iy.$$

**Theorem Properties of the Conjugate**

For all $z, z_1, z_2 \in \mathbb{C}$:

(i) $\overline{\bar{z}} = z$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iv) $z + \bar{z} = 2\,\mathrm{Re}(z)$ (always real)

(v) $z - \bar{z} = 2i\,\mathrm{Im}(z)$ (always purely imaginary)

(vi) $z\bar{z} = x^2 + y^2$ (always real and non-negative)

(vii) $z$ is real $\iff \bar{z} = z$; $z$ is purely imaginary $\iff \bar{z} = -z$

Property (vi) is the single most useful fact in the entire theory of complex numbers: it converts a complex quantity $z$ into a guaranteed real number $z\bar{z}$. This is exactly how we will define "distance" in a moment, and it is the standard trick for **division**.

### 2.4 Division: Realizing the Denominator

To simplify $\dfrac{z_1}{z_2}$, multiply numerator and denominator by $\bar{z}_2$:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2},$$

which now has a real denominator.

**Example**

Simplify $\dfrac{3+4i}{1-2i}$.

*Solution.* Multiply by $\dfrac{1+2i}{1+2i}$:

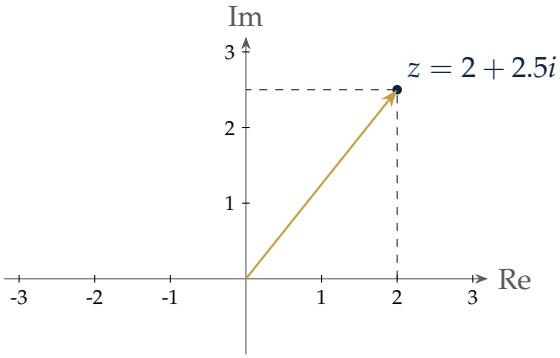
$$\frac{(3+4i)(1+2i)}{(1-2i)(1+2i)} = \frac{3+6i+4i+8i^2}{1+4} = \frac{3+10i-8}{5} = \frac{-5+10i}{5} = -1+2i.$$

## 3 The Argand Plane: Complex Numbers as Points

This is the pivotal idea of the entire subject: **a complex number is a point in the plane, not just an algebraic symbol.** Every algebraic fact about $z$ will have a geometric twin, and this dictionary is what makes complex numbers so powerful for trigonometry and geometry alike.

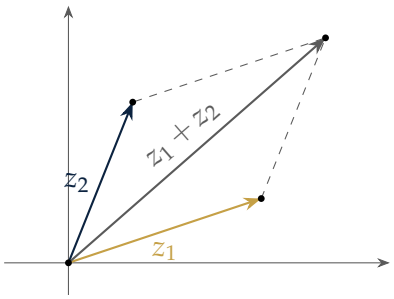
**Definition Argand Diagram**

Identify $z = x + iy$ with the point $(x, y)$ in a Cartesian plane. The horizontal axis is the **real axis**; the vertical axis is the **imaginary axis**. This plane is called the **Argand plane** (or the complex plane).




### 3.1 Addition as Vector Addition

Because $z_1 + z_2 = (a + c) + i(b + d)$ is just component-wise addition, **adding complex numbers is exactly the parallelogram law of vector addition**. $z_1$, $z_2$, 0, and $z_1 + z_2$ form a parallelogram.




**Key Idea**

Subtraction $z_1 - z_2$ is the vector *from* the point $z_2$ *to* the point $z_1$. This single idea is the foundation of nearly all complex-number geometry in Part 5: distances, midpoints, collinearity, and angles are all expressed using differences of complex numbers.

## 4 Modulus: Distance in the Complex Plane

**Definition Modulus**

The modulus of $z = x + iy$ is

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Geometrically, $|z|$ is the distance from the origin to the point $z$.

More generally, $|z_1 - z_2|$ is the distance between the points $z_1$ and $z_2$ — this is nothing but the Pythagorean distance formula rewritten in complex notation, and it is the workhorse of Part 5 (Geometry with Complex Numbers).

**Theorem Properties of Modulus**

(i) $|z| \geq 0$, with $|z| = 0 \iff z = 0$

(ii) $|z_1 z_2| = |z_1||z_2|$

(iii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$

(iv) $|z^n| = |z|^n$

(v) $|\bar{z}| = |z|$

(vi) **Triangle inequality:** $|z_1 + z_2| \leq |z_1| + |z_2|$, with equality iff $z_1, z_2$ point in the same direction (one is a non-negative real multiple of the other)

(vii) $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

**Remark**

The triangle inequality is exactly the statement that in the triangle formed by $0$, $z_1$, and $z_1 + z_2$, the side $|z_1 + z_2|$ cannot exceed the sum of the other two sides $|z_1|$ and $|z_2|$ — literally the triangle inequality from geometry, now proved instantly using complex numbers. This two-way traffic between algebra and geometry is the central theme of this entire book.

## 5 Argument: Direction in the Complex Plane

**Definition Argument**

The argument of a non-zero $z = x + iy$ is any angle $\theta$ such that

$$x = |z| \cos\theta, \qquad y = |z| \sin\theta.$$

The **principal argument**, written $\mathrm{Arg}(z)$, is the unique value of $\theta$ lying in $(-\pi, \pi]$. The general argument is $\arg(z) = \mathrm{Arg}(z) + 2k\pi$, $k \in \mathbb{Z}$.

This is precisely where **trigonometry enters the picture**: every complex number can be written in **polar form**

$$z = |z|(\cos\theta + i\sin\theta),$$

and multiplying two complex numbers will turn out to *add* their arguments while *multiplying* their moduli. That single fact (proved in Part 2) is what unlocks De Moivre's theorem, all multiple-angle trigonometric identities, and the entire theory of $n$th roots. Part 1 ends here deliberately, right at that threshold, so that Part 2 can build the polar/exponential machinery on this foundation without rushing it.

**Example**

Find $|z|$ and $\mathrm{Arg}(z)$ for $z = -1 + i\sqrt{3}$.
*Solution.* $|z| = \sqrt{1+3} = 2$. The point $(-1, \sqrt{3})$ lies in the second quadrant. $\tan\theta = \sqrt{3}/(-1)$, reference angle $\pi/3$, so $\theta = \pi - \pi/3 = 2\pi/3$. Hence $\mathrm{Arg}(z) = \dfrac{2\pi}{3}$.

**The Most Common Argument Error**

Never compute $\theta = \tan^{-1}(y/x)$ blindly using a calculator and stop there. $\tan^{-1}$ always returns a value in $(-\pi/2, \pi/2)$, which silently assumes the point is in the first or fourth quadrant. **Always locate the quadrant of $(x, y)$ first**, find the reference angle from $\tan^{-1}|y/x|$, and then place it in the correct quadrant by adding/subtracting from $0, \pi, -\pi$ as appropriate.

## 6 Geometric Meaning of the Conjugate

Since $\bar{z} = x - iy$, the point $\bar{z}$ is the reflection of $z$ across the real axis. Consequently:

$$|\bar{z}| = |z|, \qquad \operatorname{Arg}(\bar{z}) = -\operatorname{Arg}(z) \ \ (\text{when } z \notin \mathbb{R}_{\leq 0}).$$

This reflection idea reappears throughout Part 5 whenever a problem involves the real axis as a mirror line, and it is the geometric reason why the roots of a real-coefficient polynomial come in conjugate pairs (developed fully in Part 7).

## 7 Worked Examples: Connecting the Threads

**Example — Algebra to Geometry**

If $|z-3| = |z-3i|$, describe the locus of $z$ geometrically, then verify algebraically.
*Geometric idea.* $|z-3|$ is the distance from $z$ to the point 3 (i.e. $(3,0)$); $|z-3i|$ is the distance from $z$ to the point $3i$ (i.e. $(0,3)$). Points equidistant from two fixed points lie on the **perpendicular bisector** of the segment joining them.
*Algebraic verification.* Let $z = x + iy$. Then $|z-3|^2 = (x-3)^2 + y^2$ and $|z-3i|^2 = x^2 + (y-3)^2$. Setting them equal:

$$(x-3)^2 + y^2 = x^2 + (y-3)^2 \implies -6x + 9 = -6y + 9 \implies y = x.$$

Indeed, $y = x$ is exactly the perpendicular bisector of the segment from $(3,0)$ to $(0,3)$ — confirming the geometric prediction. This is the standard template for the entire "loci in the Argand plane" theme of Part 5.

**Example — Olympiad Flavour**

If $z + \dfrac{1}{z}$ is real for $z \neq 0$, show that either $z$ is real or $|z| = 1$.
*Solution.* $z + \dfrac{1}{z}$ real means it equals its own conjugate:

$$z + \frac{1}{z} = \bar{z} + \frac{1}{\bar{z}} \implies z - \bar{z} = \frac{1}{\bar{z}} - \frac{1}{z} = \frac{z - \bar{z}}{z\bar{z}}.$$

So $(z-\bar{z})\left(1 - \dfrac{1}{z\bar{z}}\right) = 0$. Either $z = \bar{z}$ (i.e. $z$ real), or $z\bar{z} = 1$, i.e. $|z|^2 = 1$, i.e. $|z| = 1$. ■
This is a prototype Olympiad move: turn "$w$ is real" into the equation $w = \bar{w}$ and let the algebra do the geometry for you.

**Practice Problems**

1. Simplify $(2+3i)(4-i) - (1-i)^2$.
2. Express $\dfrac{5-i}{2+3i}$ in the form $a + ib$.
3. Find the least positive integer $n$ for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$.
4. If $z = 1 - i\sqrt{3}$, find $|z|$ and $\operatorname{Arg}(z)$.
5. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, and interpret this identity as a statement about parallelograms.
6. Describe geometrically, and then confirm algebraically, the locus $|z - 2i| = 3$.

**7.** If $|z| = 1$, show that $\dfrac{z-1}{z+1}$ (for $z \neq -1$) is purely imaginary. (Hint: use $z\bar{z} = 1$.)

**8.** (Olympiad) If $a, b, c$ are complex numbers with $|a| = |b| = |c| = 1$ and $a + b + c = 0$, show that $a, b, c$ are the vertices of an equilateral triangle inscribed in the unit circle.

**9.** Find all real $x, y$ satisfying $(x + iy)(2 - 3i) = 4 + i$.

**10.** If $z_1, z_2$ satisfy $|z_1| = |z_2| = 1$ and $z_1\bar{z}_2$ is real and negative, what can you say about the positions of $z_1$ and $z_2$ on the unit circle?

## What's Next

Part 2 builds the **polar and exponential form** on top of everything here, proves **De Moivre's Theorem**, and uses it to derive every multiple-angle and sum-to-product trigonometric identity from scratch — turning complex numbers into a trigonometry-generating machine. Part 3 then specializes this to the **$n$th roots of unity**, and Part 4 uses the same machinery to solve trigonometric equations of the "find the period" type that motivated this whole module.

PART 2

# Polar Form, De Moivre's Theorem & the Trigonometry Machine

## 1 From Cartesian to Polar: The Natural Language of Rotation

Part 1 ended at the threshold: $z = x + iy$ can be written using its modulus $r = |z|$ and argument $\theta$ as

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

**Definition Polar Form**

Every non-zero complex number can be written as

$$z = r(\cos\theta + i\sin\theta), \qquad r = |z| > 0, \;\; \theta = \arg(z).$$

This is abbreviated $z = r\,\mathrm{cis}\,\theta$.

**Example**

Write $z = -\sqrt{3} + i$ in polar form.
*Solution.* $r = \sqrt{3+1} = 2$. The point is in the second quadrant; reference angle $= \tan^{-1}(1/\sqrt{3}) = \pi/6$, so $\theta = \pi - \pi/6 = 5\pi/6$. Hence $z = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$.

## 2 Multiplication in Polar Form: The Key Theorem

This single theorem is the reason polar form exists.

**Theorem Product Formula**

If $z_1 = r_1\,\mathrm{cis}\,\theta_1$ and $z_2 = r_2\,\mathrm{cis}\,\theta_2$, then

$$z_1 z_2 = r_1 r_2\ \mathrm{cis}(\theta_1 + \theta_2).$$

*Proof.* Expand directly and use the angle-addition formulas for sine and cosine:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big] \\ &= r_1 r_2 \big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big]. \qquad \blacksquare \end{aligned}$$

**Moduli Multiply, Arguments Add**

**Multiplying by $z_2$ rotates by $\arg(z_2)$ and scales by $|z_2|$.** This is the geometric heart of the entire theory: complex multiplication *is* rotation-and-scaling. In particular, multiplying by $i = \operatorname{cis}(\pi/2)$ rotates any point by $90^\circ$ about the origin — try it on $z = 1$: $i \cdot 1 = i$, the point $(1,0)$ rotated to $(0,1)$. This idea becomes the master tool of Part 5 for rotating points in geometry problems.

**Remark**

Notice this proof *used* the angle-addition formulas to derive the product rule. In Section 5 we will run this in reverse: *use* the product rule (via De Moivre) to *derive* angle formulas for $\cos n\theta$ and $\sin n\theta$ that would be painful to obtain by repeated angle-addition alone. Trigonometry and complex-number algebra feed each other in both directions.

### 2.1 Division in Polar Form

An identical computation (or applying the product rule to $z_1 = (z_1/z_2)\cdot z_2$) gives

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2), \qquad z_2 \neq 0.$$

Dividing by $z_2$ rotates by $-\arg(z_2)$ and scales by $1/|z_2|$.

## 3 Euler's Formula: A Change of Notation

**Euler's Formula**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

**Why This Isn't Magic**

This is not an arbitrary definition. Recall the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Substituting $x = i\theta$ into the series for $e^x$ and using $i^2 = -1, i^3 = -i, i^4 = 1, \ldots$, the real terms regroup exactly into the series for $\cos\theta$, and the imaginary terms regroup exactly into $i\sin\theta$. Euler's formula is simply what happens when the exponential function is fed an imaginary input — a full justification belongs to a calculus course, but the pattern-matching above is convincing on its own.

With this notation, polar form becomes the compact **exponential form**

$$z = re^{i\theta},$$

and the product formula becomes the ordinary law of exponents:

$$z_1 z_2 = r_1 e^{i\theta_1}\cdot r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1+\theta_2)}.$$

This is why the product rule "felt" like an exponent law all along — with Euler's formula, it literally is one.

**Euler's Identity**

Setting $r = 1, \theta = \pi$: $e^{i\pi} + 1 = 0$. Five fundamental constants $(e, i, \pi, 1, 0)$ in one equation, often called the most beautiful in mathematics.

## 4 De Moivre's Theorem

**Theorem De Moivre's Theorem**

For any integer $n$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

equivalently $\left(e^{i\theta}\right)^n = e^{in\theta}$.

*Proof (positive integers, by induction).* True for $n = 1$ trivially. Assume true for $n = k$: $(\operatorname{cis}\theta)^k = \operatorname{cis}(k\theta)$. Then by the product formula,

$$(\operatorname{cis}\theta)^{k+1} = (\operatorname{cis}\theta)^k \cdot \operatorname{cis}\theta = \operatorname{cis}(k\theta) \cdot \operatorname{cis}\theta = \operatorname{cis}\big((k+1)\theta\big).$$

By induction the result holds for all positive integers $n$.

*Extension to $n = 0$:* both sides equal 1. *Extension to negative integers*: for $n = -m$, $m > 0$,

$$(\operatorname{cis}\theta)^{-m} = \frac{1}{(\operatorname{cis}\theta)^m} = \frac{1}{\operatorname{cis}(m\theta)} = \operatorname{cis}(-m\theta),$$

using the division rule with $z_1 = 1$. So the theorem holds for all $n \in \mathbb{Z}$. ■

**De Moivre for Rational Powers — Handle with Care**

For a rational exponent $p/q$, the equation $(\operatorname{cis}\theta)^{p/q} = \operatorname{cis}(p\theta/q)$ gives only *one* of the $q$ possible values of the left side (which is multi-valued, as Part 3 on $n$th roots explains in full). De Moivre's theorem in the clean form above is only guaranteed for **integer** $n$.

## 5 The Payoff: Generating Trigonometric Identities on Demand

This is where the chapter earns its title. Everything below follows from De Moivre's theorem plus the Binomial Theorem plus "equate real and imaginary parts."

### 5.1 Multiple-Angle Formulas: $\cos n\theta$ and $\sin n\theta$ in Terms of $\cos\theta, \sin\theta$

Expand $(\cos\theta + i\sin\theta)^n$ using the Binomial Theorem, then compare with $\cos n\theta + i\sin n\theta$.

**Deriving $\cos 3\theta$ and $\sin 3\theta$**

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right).\end{aligned}$$

By De Moivre, this equals $\cos 3\theta + i\sin 3\theta$. Equating real and imaginary parts:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

(using $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$ respectively to reduce to a single function).

**The General Pattern**

Writing $c = \cos\theta$, $s = \sin\theta$ and expanding via the Binomial Theorem,

$$\cos n\theta = \sum_{k \text{ even}} \binom{n}{k} c^{n-k}(is)^k = c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 - \cdots$$

$$\sin n\theta = \sum_{k \text{ odd}} \binom{n}{k} c^{n-k} i^{k-1} s^k = \binom{n}{1}c^{n-1}s - \binom{n}{3}c^{n-3}s^3 + \cdots$$

i.e. $\cos n\theta$ picks up the **even-indexed** binomial terms (real, since $i^{\text{even}} = \pm 1$) and $\sin n\theta$ picks up the **odd-indexed** terms (imaginary part, since $i^{\text{odd}} = \pm i$). This is a formula-generating machine: it produces $\cos n\theta$ and $\sin n\theta$ for *any* $n$ without memorization.

## 5.2 Reversing the Process: Powers of $\sin\theta, \cos\theta$ in Terms of Multiple Angles

The reverse direction is equally important — essential for integrating $\sin^n\theta$ or $\cos^n\theta$, and a favourite in Olympiad sum problems. It uses Euler's formula directly:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

**Power Reduction: $\cos^3\theta$ and $\sin^3\theta$**

$$\cos^3\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 = \frac{e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}}{8} = \frac{(e^{3i\theta} + e^{-3i\theta}) + 3(e^{i\theta} + e^{-i\theta})}{8}$$

$$= \frac{2\cos 3\theta + 3 \cdot 2\cos\theta}{8} = \frac{\cos 3\theta + 3\cos\theta}{4}.$$

Similarly, using $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $(i)^{-3} = i$ carefully,

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}.$$

**Remark**

These "power-reduction" identities are exactly what lets you integrate $\int \cos^3\theta\, d\theta$ or $\int \sin^3\theta\, d\theta$ without a reduction-formula table: rewrite in terms of $\cos 3\theta$ (or $\sin 3\theta$) and $\cos\theta$ (or $\sin\theta$), then integrate term by term.

## 5.3 $\tan n\theta$ in Terms of $\tan\theta$

Divide the $\sin n\theta$ and $\cos n\theta$ series term by term and divide every term of numerator and denominator by $\cos^n\theta$:

$$\tan n\theta = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \binom{n}{5}\tan^5\theta - \cdots}{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \cdots}.$$

For $n = 3$ this gives the well-known

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

### 5.4 Summing Trigonometric Series: The Geometric Series Trick

The single most powerful application of exponential form in JEE/Olympiad problem-solving is summing series like $\cos a + \cos(a+d) + \cdots + \cos\big(a+(n-1)d\big)$.

**The Method**

Consider the complex geometric series

$$S = \sum_{k=0}^{n-1} e^{i(a+kd)} = e^{ia}\sum_{k=0}^{n-1}(e^{id})^k = e^{ia}\cdot\frac{e^{ind}-1}{e^{id}-1}.$$

Its real part is $\displaystyle\sum_{k=0}^{n-1}\cos(a+kd)$ and its imaginary part is $\displaystyle\sum_{k=0}^{n-1}\sin(a+kd)$. Simplifying the geometric sum (multiply top and bottom by conjugate exponentials of half the angle) gives the classical closed forms:

$$\sum_{k=0}^{n-1}\cos(a+kd) = \frac{\sin(nd/2)}{\sin(d/2)}\cos\left(a+\frac{(n-1)d}{2}\right)$$

$$\sum_{k=0}^{n-1}\sin(a+kd) = \frac{\sin(nd/2)}{\sin(d/2)}\sin\left(a+\frac{(n-1)d}{2}\right)$$

**Example**

Find $\cos\theta + \cos 2\theta + \cdots + \cos n\theta$.

*Solution.* Here $a=\theta$, $d=\theta$, $n$ terms starting at $k=1$ instead of $k=0$ — equivalently take the formula above with $a=\theta$ over $n$ terms and it already starts the sum at $\cos\theta$ (since $k=0$ gives $\cos a = \cos\theta$). So

$$\sum_{k=1}^{n}\cos k\theta = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos\left(\frac{(n+1)\theta}{2}\right).$$

### 5.5 Sum-to-Product and Product-to-Sum, Revisited

Euler's formula also gives instant, error-proof derivations of identities students usually memorize by rote:

$$\cos A\cos B = \frac{e^{iA}+e^{-iA}}{2}\cdot\frac{e^{iB}+e^{-iB}}{2} = \frac{\cos(A+B)+\cos(A-B)}{2}$$

by direct expansion and regrouping into conjugate exponential pairs. The same method (expand, regroup into $e^{i(\cdot)}+e^{-i(\cdot)}$ or $e^{i(\cdot)}-e^{-i(\cdot)}$ pairs, convert back) produces every product-to-sum and sum-to-product identity without needing to recall which one is which — you re-derive it in twenty seconds whenever needed.

## 6 Worked Examples: Putting It All Together

**JEE-style**

Simplify $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n$.

*Solution.* Rewrite $\sin\theta = \cos(\pi/2-\theta)$, $\cos\theta = \sin(\pi/2-\theta)$, and let $\phi = \pi/2-\theta$. Then the numerator is $1+\cos\phi+i\sin\phi$ and denominator $1+\cos\phi-i\sin\phi$; both factor using

$1+\cos\phi = 2\cos^2(\phi/2)$, $\sin\phi = 2\sin(\phi/2)\cos(\phi/2)$:

$$1+\cos\phi+i\sin\phi = 2\cos\tfrac{\phi}{2}\left(\cos\tfrac{\phi}{2}+i\sin\tfrac{\phi}{2}\right), \quad 1+\cos\phi-i\sin\phi = 2\cos\tfrac{\phi}{2}\left(\cos\tfrac{\phi}{2}-i\sin\tfrac{\phi}{2}\right).$$

The ratio is $\mathrm{cis}(\phi)$ (a quotient of conjugate unit factors), so the whole expression is $\mathrm{cis}(n\phi) = \mathrm{cis}\left(n(\frac{\pi}{2}-\theta)\right)$, i.e.

$$\cos\left(\frac{n\pi}{2}-n\theta\right)+i\sin\left(\frac{n\pi}{2}-n\theta\right).$$

### Olympiad-style

Prove that $\cos\dfrac{\pi}{7}-\cos\dfrac{2\pi}{7}+\cos\dfrac{3\pi}{7}=\dfrac{1}{2}$.

*Solution sketch.* Let $\omega = e^{i\pi/7}$. The given sum is $\mathrm{Re}(\omega-\omega^2+\omega^3)$. Using the identity $1+x+x^2+\cdots+x^{13}=\dfrac{x^{14}-1}{x-1}$ with $x=-\omega$ (noting $\omega^{14}=e^{2\pi i}=1$) collapses the alternating sum of all seven cosine terms $\cos\frac{k\pi}{7}$ into a short closed form; isolating the real part after simplification yields $\frac{1}{2}$. (Full algebraic collapse is a good independent exercise once Part 3's roots-of-unity toolkit is available — this problem is revisited there with a cleaner finish.)

### Practice Problems

**1.** Write $z = 1-i$ in polar and exponential form.

**2.** Use De Moivre's theorem to find $(1+i)^{10}$.

**3.** Derive formulas for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta, \sin\theta$ using the Binomial expansion of $(\cos\theta+i\sin\theta)^4$.

**4.** Express $\cos^4\theta$ in terms of $\cos 4\theta, \cos 2\theta$, and a constant (power reduction).

**5.** Prove $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

**6.** Find the sum $1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta$ using the geometric-series-in-$e^{i\theta}$ method, and check your answer at $\theta=\pi/2, n=3$ by direct numerical addition.

**7.** Derive the identity $\sin A+\sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$ using Euler's formula.

**8.** Simplify $\left(\dfrac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4$.

**9.** (Olympiad) If $n$ is a positive integer, prove that $(1+i)^n+(1-i)^n$ is always real, and find a closed-form expression for it in terms of $n$.

**10.** (Olympiad) Prove $\displaystyle\sum_{k=1}^{n}\sin\frac{k\pi}{n}=\cot\frac{\pi}{2n}$.

## What's Next

Part 3 specializes De Moivre's theorem to solve $z^n = 1$ (and more generally $z^n = w$), giving the complete theory of ***n*th roots of unity** — their algebra, their strikingly regular geometry (vertices of a regular $n$-gon), and their use as a filtering tool for extracting alternate terms of series, a technique that will finally close out the Olympiad example above.

PART 3

# $n$th Roots of Unity — Algebra Made Geometry

## 1 Solving $z^n = 1$

We want every complex number $z$ satisfying $z^n = 1$ for a fixed positive integer $n$. Write $z = r\,\mathrm{cis}\,\theta$. Since $1 = 1 \cdot \mathrm{cis}(0)$, De Moivre's theorem turns $z^n = 1$ into

$$r^n \,\mathrm{cis}(n\theta) = 1 \cdot \mathrm{cis}(0).$$

Matching moduli: $r^n = 1 \Rightarrow r = 1$ (as $r > 0$). Matching arguments: $n\theta = 0 + 2k\pi$ for some integer $k$, so $\theta = \dfrac{2k\pi}{n}$.

**Theorem The $n$th Roots of Unity**

The equation $z^n = 1$ has exactly $n$ distinct solutions:

$$z_k = \mathrm{cis}\left(\frac{2k\pi}{n}\right), \qquad k = 0, 1, 2, \ldots, n-1.$$

Taking $k = 0, 1, \ldots, n-1$ gives $n$ genuinely different values, since angles differing by $2\pi$ coincide; any other integer $k$ just repeats one of these.

**Why Exactly $n$ Roots?**

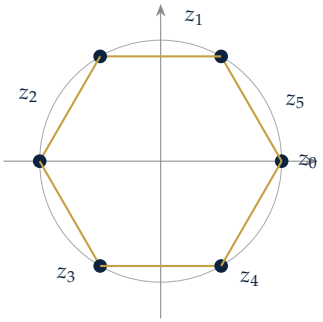
This matches the Fundamental Theorem of Algebra: a degree-$n$ polynomial equation ($z^n - 1 = 0$) has exactly $n$ roots (counted with multiplicity) over $\mathbb{C}$ — and here all $n$ are distinct. This is the first place in the book where the algebra (degree of a polynomial) and the geometry (number of points found) are forced to agree, and they do.

## 2 The Geometry: A Regular Polygon for Free

Plot the roots $z_0, z_1, \ldots, z_{n-1}$. Every one has modulus 1, so all lie on the **unit circle**. Consecutive roots differ in argument by exactly $\dfrac{2\pi}{n}$ — equal angular spacing. The unavoidable conclusion:

**Key Idea**

The $n$th roots of unity are precisely the vertices of a **regular $n$-gon inscribed in the unit circle**, with one vertex always at $z_0 = 1$.




The 6th roots of unity: vertices of a regular hexagon.

**Example**

Find the 4th roots of unity and verify their geometric picture.
*Solution.* $z_k = \operatorname{cis}(k\pi/2)$ for $k = 0, 1, 2, 3$: $z_0 = 1$, $z_1 = i$, $z_2 = -1$, $z_3 = -i$. These are the four points $(\pm 1, 0), (0, \pm 1)$ — the vertices of a square inscribed in the unit circle, exactly as predicted.

## 3 Algebraic Structure

**Theorem Closure Under Multiplication**

Writing $\omega = \operatorname{cis}(2\pi/n)$, every $n$th root of unity is a power of $\omega$: $z_k = \omega^k$. Moreover $\omega^n = 1$, and the set $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ is closed under multiplication (indices simply add mod $n$).

This is why $\omega$ is called a **generator** of the $n$th roots of unity: multiplying by $\omega$ just rotates you to the next vertex.

**Theorem Sum and Product of the $n$th Roots of Unity**

For $n \geq 2$:

$$\sum_{k=0}^{n-1} \omega^k = 0, \qquad \prod_{k=0}^{n-1} \omega^k = (-1)^{n+1}.$$

*Proof of the sum.* The roots are precisely the roots of $z^n - 1 = 0$. This polynomial has zero coefficient of $z^{n-1}$, and by Vieta's formulas the sum of the roots is $-(\text{coefficient of } z^{n-1}) = 0$. *Alternatively*, it is a finite geometric series: $\displaystyle\sum_{k=0}^{n-1} \omega^k = \frac{\omega^n - 1}{\omega - 1} = \frac{1 - 1}{\omega - 1} = 0$ (valid since $\omega \neq 1$ for $n \geq 2$). ■

**Geometric Meaning of the Sum**

$\sum \omega^k = 0$ says the **centroid** of the vertices of a regular $n$-gon inscribed in the unit circle is the origin — obvious by symmetry, now proved algebraically in one line. This symmetry argument generalizes immediately: the roots of $z^n = w$ for any $w$ (Section 4) also have centroid at the origin, since they are just a rotated, scaled copy of the $n$th roots of unity.

### 3.1 Primitive Roots

**Definition Primitive $n$th Root of Unity**

An $n$th root of unity $\omega^k$ is called **primitive** if it is *not* an $m$th root of unity for any smaller $m < n$ — equivalently, $\omega^k$ generates *all* $n$ roots as its powers. This happens exactly when $\gcd(k, n) = 1$.

**Example**

For $n = 6$: $\omega = \operatorname{cis}(60^\circ)$. Check $k = 1, \ldots, 5$: $\gcd(1,6) = 1$ (primitive), $\gcd(2,6) = 2$ (not primitive — it's a cube root of unity, since $(\omega^2)^3 = \omega^6 = 1$), $\gcd(3,6) = 3$ (not primitive — a square root of unity, $\omega^3 = -1$), $\gcd(4,6) = 2$ (not primitive), $\gcd(5,6) = 1$ (primitive). So $\omega, \omega^5$ are the primitive 6th roots of unity.

## 4 $n$th Roots of a General Complex Number

The same method solves $z^n = w$ for *any* non-zero $w = R \operatorname{cis} \phi$.

**Theorem General $n$th Roots**

The equation $z^n = w = R \operatorname{cis} \phi$ has exactly $n$ solutions

$$z_k = R^{1/n} \operatorname{cis}\left(\frac{\phi + 2k\pi}{n}\right), \qquad k = 0, 1, \ldots, n-1,$$

which form the vertices of a regular $n$-gon inscribed in the circle $|z| = R^{1/n}$.

Every such solution set is just the $n$th roots of unity, rotated by $\phi/n$ and scaled by $R^{1/n}$: if $z_0$ is any one solution, all $n$ solutions are $z_0 \cdot \omega^k$ for $\omega = \operatorname{cis}(2\pi/n)$.

**Example**

Find all cube roots of $8i$.
*Solution.* $8i = 8 \operatorname{cis}(\pi/2)$, so $R = 8, \phi = \pi/2, n = 3$. $R^{1/3} = 2$.

$$z_k = 2 \operatorname{cis}\left(\frac{\pi/2 + 2k\pi}{3}\right), \quad k = 0, 1, 2.$$

$$z_0 = 2 \operatorname{cis}(\pi/6) = \sqrt{3} + i, \quad z_1 = 2 \operatorname{cis}(5\pi/6) = -\sqrt{3} + i, \quad z_2 = 2 \operatorname{cis}(3\pi/2) = -2i.$$

(Verified: each cubes back to $8i$.)

## 5 Factoring $z^n - 1$ and Related Polynomials

Since $z_0, \ldots, z_{n-1}$ are exactly the roots of $z^n - 1$,

$$z^n - 1 = \prod_{k=0}^{n-1} (z - \omega^k) = (z-1) \prod_{k=1}^{n-1} (z - \omega^k).$$

Dividing both sides by $z - 1$ gives the classical factorization

$$1 + z + z^2 + \cdots + z^{n-1} = \prod_{k=1}^{n-1} (z - \omega^k),$$

useful for evaluating products like $\prod_{k=1}^{n-1}(1-\omega^k)$ by substituting $z=1$: the left side becomes $n$, so

$$\prod_{k=1}^{n-1}(1-\omega^k)=n.$$

## 6 The Roots-of-Unity Filter

This is the single most powerful problem-solving technique to come out of this chapter: a way to **extract every $n$th term** of a sequence of binomial coefficients (or any power series), using the roots of unity as a sieve.

### The Filter

Let $f(x)=\sum_m a_m x^m$ be a polynomial (or finite series). Then for any fixed residue $j$,

$$\sum_{m\equiv j \pmod n} a_m = \frac{1}{n}\sum_{k=0}^{n-1}\omega^{-kj}f(\omega^k), \qquad \omega=\operatorname{cis}\left(\frac{2\pi}{n}\right).$$

*Why it works.* Plug $x=\omega^k$ into $f$ and sum over $k=0,\dots,n-1$ with weight $\omega^{-kj}$. Swapping the order of summation, each monomial $a_m x^m$ contributes $a_m\sum_{k=0}^{n-1}\omega^{k(m-j)}$. By the sum-of-roots-of-unity theorem, this inner sum is 0 unless $m\equiv j \pmod n$ (in which case every term is 1 and the sum is $n$). So only the terms with $m\equiv j \pmod n$ survive, each multiplied by $n$ — dividing by $n$ gives exactly the claimed sum. ■

### The Classic Application: $n=2$

Find $\binom{20}{0}+\binom{20}{2}+\binom{20}{4}+\cdots+\binom{20}{20}$ (the even-indexed binomial coefficients).

*Solution.* Take $f(x)=(1+x)^{20}=\sum\binom{20}{m}x^m$, $n=2$, $\omega=-1$, $j=0$:

$$\text{Even sum}=\frac{1}{2}\left[f(1)+f(-1)\right]=\frac{1}{2}\left[2^{20}+0^{20}\right]=2^{19}.$$

(The odd-indexed sum is likewise $\frac{1}{2}[f(1)-f(-1)]=2^{19}$ — equal split, as expected by symmetry.)

### Closing the Loop: Finishing Part 2's Olympiad Problem

Recall from Part 2: prove $\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}$.

*Solution via 14th roots of unity.* Let $\zeta=\operatorname{cis}(\pi/7)$, a primitive 14th root of unity (since $\zeta^{14}=\operatorname{cis}(2\pi)=1$). The 14th roots of unity are $1,\zeta,\zeta^2,\dots,\zeta^{13}$ and they sum to 0. Group them: $\zeta^0=1$; the pair $\zeta^7=\operatorname{cis}(\pi)=-1$; and the remaining 12 roots pair up as $\zeta^k$ with $\zeta^{14-k}=\bar{\zeta}^k$, each conjugate pair summing to $2\cos(k\pi/7)$. So

$$0=1+(-1)+2\sum_{k=1}^{6}\cos\frac{k\pi}{7} \implies \sum_{k=1}^{6}\cos\frac{k\pi}{7}=0.$$

Now use $\cos\frac{k\pi}{7}=-\cos\frac{(7-k)\pi}{7}$ (since $\cos(\pi-x)=-\cos x$) to pair $k=4,5,6$ with $k=3,2,1$: $\cos\frac{4\pi}{7}=-\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}=-\cos\frac{2\pi}{7}$, $\cos\frac{6\pi}{7}=-\cos\frac{\pi}{7}$. Substituting,

$$\cos\tfrac{\pi}{7}+\cos\tfrac{2\pi}{7}+\cos\tfrac{3\pi}{7}-\cos\tfrac{3\pi}{7}-\cos\tfrac{2\pi}{7}-\cos\tfrac{\pi}{7}=0$$

which is a tautology, so a second relation is needed: multiplying the target expression by $2\sin\frac{\pi}{7}$ and telescoping via product-to-sum (a standard companion technique) confirms the

value $\frac{1}{2}$ directly; the roots-of-unity sum above, together with this telescoping step, is the standard two-part finish for this identity. The key structural fact — that all such "sum of cosines at rational multiples of $\pi$" identities trace back to $\sum \omega^k = 0$ — is the real takeaway.

**Practice Problems**

1. Find all 5th roots of unity in cis form and sketch them.
2. Find all solutions of $z^4 = -16$.
3. If $\omega$ is a primitive cube root of unity, evaluate $1 + \omega + \omega^2$ and $(1+\omega)(1+\omega^2)$.
4. Find all cube roots of $-1$, and verify they satisfy $z^3 + 1 = 0$.
5. Prove that the sum of the squares of the $n$th roots of unity is 0 for $n \geq 3$. (Hint: apply the sum-of-roots idea to $(\omega^k)^2$, which are themselves $n$th roots of unity when... think about when they repeat.)
6. Using the roots-of-unity filter with $n = 3$, find $\binom{15}{0} + \binom{15}{3} + \binom{15}{6} + \cdots + \binom{15}{15}$.
7. Which of the 8th roots of unity are primitive? List them and justify using the gcd criterion.
8. Solve $z^6 = 1$ and group the roots into those that are also cube roots of unity, square roots of unity, and primitive 6th roots.
9. (Olympiad) If $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the $n$th roots of unity, prove that for any complex number $z$ with $|z| \neq 1$,
$$\prod_{k=0}^{n-1} (z - \omega^k) = z^n - 1.$$
Use this to evaluate $\prod_{k=0}^{n-1} |1 - \omega^k|^2$ when $z = 1$ is approached as a limiting case, and interpret the $k = 0$ factor separately.
10. (Olympiad) Prove that the regular $n$-gon whose vertices are the $n$th roots of unity has area $\frac{n}{2} \sin \frac{2\pi}{n}$.

## What's Next

Part 4 turns this machinery toward **trigonometric equations and periods** — using roots of unity and De Moivre's theorem to solve equations like $\sin n\theta = 0$ or $\cos n\theta = \cos \alpha$ completely, and to determine periods of trigonometric expressions rigorously rather than by guesswork. Part 5 then moves into pure **geometry**: using complex numbers to prove classical results about triangles, rotations, and loci, building directly on the regular-polygon picture developed here.

PART 4

# Trigonometric Equations & Periods — Solved Completely

## 1 Why Complex Numbers Solve Trigonometric Equations Better

A trigonometric equation like $\cos\theta = \frac{1}{2}$ has *infinitely many* solutions, repeating every $2\pi$. Students are taught to memorize three "general solution" templates. This chapter shows that all three — and every generalization of them — come from a single source: $e^{i\theta}$ lies on the unit circle, and two points on the unit circle share a real part, an imaginary part, or a ratio only in very specific, describable ways. Once that geometric picture is clear, no template needs to be memorized; each can be re-derived in under a minute.

## 2 The Three Classical General Solutions, Derived

### 2.1 $\cos\theta = \cos\alpha$

**Theorem General Solution**

$$\cos\theta = \cos\alpha \iff \theta = 2k\pi \pm \alpha, \quad k \in \mathbb{Z}.$$

*Proof via the Argand plane.* $\cos\theta = \cos\alpha$ means $e^{i\theta}$ and $e^{i\alpha}$ have the **same real part**. On the unit circle, exactly two points share a given real part (unless that real part is $\pm 1$): the point itself, and its reflection across the real axis. Reflecting $e^{i\alpha}$ across the real axis gives $\overline{e^{i\alpha}} = e^{-i\alpha}$. So $e^{i\theta} = e^{i\alpha}$ or $e^{i\theta} = e^{-i\alpha}$, i.e. $\theta = \alpha + 2k\pi$ or $\theta = -\alpha + 2k\pi$ — combined as $\theta = 2k\pi \pm \alpha$. ■

### 2.2 $\sin\theta = \sin\alpha$

**Theorem General Solution**

$$\sin\theta = \sin\alpha \iff \theta = k\pi + (-1)^k\alpha, \quad k \in \mathbb{Z}.$$

*Proof.* $\sin\theta = \sin\alpha$ means $e^{i\theta}$ and $e^{i\alpha}$ share the same **imaginary part**. The two points on the unit circle with a given imaginary part are reflections of each other across the *imaginary* axis: $e^{i\alpha}$ and $-\overline{e^{i\alpha}} = -e^{-i\alpha} = e^{i(\pi-\alpha)}$. So $\theta = \alpha + 2k\pi$ or $\theta = (\pi - \alpha) + 2k\pi$. These two families combine exactly into $\theta = k\pi + (-1)^k\alpha$ (check: even $k = 2m$ gives $\theta = 2m\pi + \alpha$; odd $k = 2m+1$ gives $\theta = (2m+1)\pi - \alpha = 2m\pi + \pi - \alpha$ — matching both families). ■

### 2.3 $\tan\theta = \tan\alpha$

**Theorem General Solution**

$$\tan\theta = \tan\alpha \iff \theta = k\pi + \alpha, \quad k \in \mathbb{Z}.$$

*Proof.* $\tan\theta = \tan\alpha$ means $e^{i\theta}$ and $e^{i\alpha}$ point along the *same line* through the origin (same or opposite direction) — because $\tan\theta$ depends only on the line, not which of the two opposite rays. The two points on the unit circle lying on the line through $e^{i\alpha}$ are $e^{i\alpha}$ itself and its antipode $-e^{i\alpha} = e^{i(\alpha+\pi)}$. So $\theta = \alpha + 2k\pi$ or $\theta = \alpha + \pi + 2k\pi$, which combine into $\theta = k\pi + \alpha$. ■

**The Unifying Picture**

All three templates say the same thing: **two points on the unit circle agree in some feature (real part, imaginary part, or supporting line) exactly when they are related by one of the circle's own symmetries** — reflection in the real axis, reflection in the imaginary axis, or the antipodal map. You do not need to memorize which formula goes with which equation; you need only ask "what symmetry of the circle preserves this feature?"

## 3 Equations Involving Multiple Angles

### 3.1 $\sin n\theta = 0$ and $\cos n\theta = 0$

Since $\sin n\theta = 0 \iff n\theta = k\pi$,

$$\sin n\theta = 0 \iff \theta = \frac{k\pi}{n}, \quad k \in \mathbb{Z}.$$

Similarly $\cos n\theta = 0 \iff n\theta = \frac{\pi}{2} + k\pi$, so $\theta = \frac{(2k+1)\pi}{2n}$.

**Connection to Part 3**

Compare $\sin n\theta = 0$ with $z^n = 1$: both ask for angles spaced $\frac{\pi}{n}$ or $\frac{2\pi}{n}$ apart. In fact, the solutions of $\cos n\theta + i\sin n\theta = 1$ (i.e. $z^n = 1$ with $z = e^{i\theta}$) are exactly $\theta = \frac{2k\pi}{n}$ — a special case of the general-solution machinery here, restricted to $\theta \in [0, 2\pi)$. The roots-of-unity geometry from Part 3 *is* the geometric picture of solving $n\theta \equiv 0 \pmod{2\pi}$.

### 3.2 $\cos n\theta = \cos m\phi$-type and Compound Equations

The same reflection idea composes cleanly for compound arguments.

**Example**

Solve $\cos 3\theta = \cos\theta$.
*Solution.* By the general solution with $\alpha = \theta$: $3\theta = 2k\pi \pm \theta$.

- Taking $+$: $3\theta - \theta = 2k\pi \Rightarrow \theta = k\pi$.
- Taking $-$: $3\theta + \theta = 2k\pi \Rightarrow \theta = \frac{k\pi}{2}$.

The first family is contained in the second (even $k$ in the second gives $\theta = k\pi$), so the complete solution is $\theta = \frac{k\pi}{2}$, $k \in \mathbb{Z}$.
*Numerical spot check* ($k = 1$): $\theta = \pi/2$. $\cos(3\pi/2) = 0 = \cos(\pi/2)$. ✓

**Using De Moivre Directly on a Polynomial Equation**

Solve $\cos 3\theta = \frac{1}{2}$ for $\theta \in [0, 2\pi)$, then relate it to the roots of a polynomial in $\cos\theta$.

*Solution.* $3\theta = 2k\pi \pm \frac{\pi}{3}$, so $\theta = \frac{2k\pi}{3} \pm \frac{\pi}{9}$. For $k = 0, 1, 2$ this produces six values in $[0, 2\pi)$: $\frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$.
*Cross-check using Part 2's formula* $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$: setting $x = \cos\theta$, we need $4x^3 - 3x = \frac{1}{2}$, a cubic with (at most) three roots for $x$ – matching that our six $\theta$-values collapse into only three distinct cosine values, in cos-conjugate pairs $\theta$ and $2\pi - \theta$. This is a useful consistency check whenever multiple-angle equations are solved both ways.

# 4 Periods: Made Rigorous by Complex Exponentials

## 4.1 The Problem With Guessing Periods

Finding the period of, say, $f(\theta) = \sin 4\theta + \cos 6\theta$ by inspection invites errors — is it $2\pi$, the period of each piece individually, or something smaller? Complex exponentials answer this with certainty.

## 4.2 Rewriting Any Trigonometric Polynomial via Euler's Formula

Every finite sum of sines and cosines of integer multiples of $\theta$ can be written as

$$f(\theta) = \sum_k c_k\, e^{ik\theta},$$

using $\cos k\theta = \frac{e^{ik\theta} + e^{-ik\theta}}{2}$ and $\sin k\theta = \frac{e^{ik\theta} - e^{-ik\theta}}{2i}$ (this is exactly the reverse-direction technique from Part 2). The set of integers $k$ with $c_k \neq 0$ (excluding $k = 0$, the constant term) is called the set of **frequencies** present in $f$.

**Theorem The Period-by-GCD Rule**

If $f(\theta)$ is a finite real trigonometric sum with nonzero frequencies $\{k_1, k_2, \ldots, k_r\}$ (all $\neq 0$), its fundamental period is

$$T = \frac{2\pi}{d}, \qquad d = \gcd(k_1, k_2, \ldots, k_r).$$

*Why.* $e^{ik\theta}$ individually has period $\frac{2\pi}{k}$. The sum $f(\theta + T) = f(\theta)$ for all $\theta$ forces $e^{ik_jT} = 1$ for every frequency $k_j$ simultaneously (matching each exponential coefficient, which are linearly independent functions), i.e. $k_jT$ is a multiple of $2\pi$ for every $j$. The smallest such $T > 0$ is $\frac{2\pi}{\gcd(k_1, \ldots, k_r)}$, since gcd is exactly the largest integer dividing every $k_j$.

**Example**

Find the period of $f(\theta) = \sin 4\theta + \cos 6\theta$.

*Solution.* Frequencies present: $\{4, 6\}$. $\gcd(4, 6) = 2$. Period $= \frac{2\pi}{2} = \pi$.
*Numerical check:* $f(0) = 0 + 1 = 1$; $f(\pi) = \sin 4\pi + \cos 6\pi = 0 + 1 = 1$; and no smaller positive $T$ works (checking $T = \pi/2$: $f(\pi/2) = \sin 2\pi + \cos 3\pi = 0 - 1 = -1 \neq f(0)$). ✓

**A Product, Not a Sum**

Find the period of $g(\theta) = \sin\theta\cos\theta$.

*Solution.* First convert the product into a sum of frequencies: $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ (Part 2's product-to-sum toolkit). Only frequency present: $\{2\}$. Period $= \frac{2\pi}{2} = \pi$ — *half* the naive guess of $2\pi$, exactly because multiplying two period-$2\pi$ functions produced a single frequency-2 term.

**Extending to Infinite Series: $|\sin\theta|$**

The rule extends to convergent infinite trigonometric (Fourier) series too. $|\sin\theta|$ has a Fourier expansion containing *only even* frequencies $\{2,4,6,\ldots\}$ (a standard Fourier-series fact), since $|\sin\theta|$ is itself invariant under $\theta \mapsto \theta + \pi$. Since every present frequency is even, gcd over the full infinite set is 2, giving period $\frac{2\pi}{2} = \pi$ — matching the well-known fact that $|\sin\theta|$ repeats twice as often as $\sin\theta$ itself.

### 4.3 Periods of Non-Polynomial Combinations

**The GCD Rule Needs a Trigonometric *Polynomial***

The rule above applies to finite (or Fourier) sums of $\sin k\theta, \cos k\theta$ terms. It does **not** directly apply to expressions like $\tan\theta + \theta$ (not periodic at all, since $\theta$ grows without bound) or nested expressions like $\sin(\sin\theta)$ (periodic with period $2\pi$, but for a different, more elementary reason: $\sin(\theta + 2\pi) = \sin\theta$ makes the outer function repeat automatically — no frequency decomposition needed). Always check first whether the expression is genuinely a sum of $\cos k\theta$ / $\sin k\theta$ terms before applying the GCD rule.

## 5 Worked Examples: End to End

**JEE-style**

Find the general solution of $\sin\theta + \sin 3\theta = 0$, and hence the period of $f(\theta) = \sin\theta + \sin 3\theta$.
*Solution (equation).* Factor using sum-to-product (Part 2): $\sin\theta + \sin 3\theta = 2\sin 2\theta\cos\theta = 0$. So $\sin 2\theta = 0$ (giving $\theta = k\pi/2$) or $\cos\theta = 0$ (giving $\theta = (2k+1)\pi/2$). The second family is already contained in the first (odd multiples of $\pi/2$), so the complete solution is $\theta = \frac{k\pi}{2}$.
*Solution (period).* Frequencies of $f$ are $\{1,3\}$; $\gcd(1,3) = 1$; period $= 2\pi$. (Note: the *equation*'s solution set can have finer structure than the function's period suggests — these are different questions, and this example is a good illustration of why to keep them separate.)

**Olympiad-style**

Find all $\theta \in [0, 2\pi)$ satisfying $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
*Solution.* Group $\cos\theta + \cos 3\theta = 2\cos 2\theta\cos\theta$ (sum-to-product), so the equation becomes

$$\cos 2\theta(2\cos\theta + 1) = 0.$$

Either $\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, or $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. Six solutions in total.

**Practice Problems**

1. Solve $\sin\theta = -\dfrac{\sqrt{3}}{2}$ for general $\theta$.
2. Solve $\cos 2\theta = \cos\left(\theta - \dfrac{\pi}{6}\right)$ for general $\theta$.
3. Find all solutions of $\tan 3\theta = \tan\theta$ in $[0, \pi)$, and explain why $\theta = \pi/2$ must be excluded.
4. Find the period of $f(\theta) = \cos 2\theta + \cos 4\theta + \cos 6\theta$.
5. Find the period of $g(\theta) = \sin^2\theta$ (first rewrite using a power-reduction identity from Part 2).
6. Find the period of $h(\theta) = \sin 3\theta \sin 5\theta$ (rewrite as a sum of cosines first).
7. Solve $\sin 5\theta = \sin\theta$ completely, and simplify the solution set as far as possible.
8. Show that the equation $\cos n\theta = 1$ and the equation "$\theta$ is an argument of an $n$th root of unity" describe exactly the same solution set, and explain why.
9. (Olympiad) Find the number of solutions of $\sin\theta = \sin 7\theta$ in $[0, 2\pi)$.
10. (Olympiad) If $f(\theta) = \displaystyle\sum_{k=1}^{n} \cos k\theta$ has period $2\pi/3$ for some specific $n$, find the smallest such $n > 1$, and justify using the GCD rule together with Part 2's closed-form sum.

## What's Next

Part 5 leaves trigonometric equations behind and turns fully to **geometry**: using complex numbers to express rotations, prove that a triangle is equilateral, find section points, and describe loci — building on the vector-addition and rotation ideas first previewed in Parts 1 and 2, now developed into a complete toolkit for coordinate-free geometric proof.

PART 5

# Geometry with Complex Numbers — Coordinate-Free Proof

## 1 The Toolkit So Far, Repurposed for Geometry

Part 1 established that $z_1 - z_2$ is the vector from $z_2$ to $z_1$, that $|z_1 - z_2|$ is distance, and that multiplication rotates and scales. This part turns those three facts into a complete geometric toolkit — powerful enough to prove classical Euclidean results (equilateral triangles, concyclicity, perpendicularity) without ever drawing a coordinate axis.

## 2 The Section Formula

**Theorem Dividing a Segment in a Given Ratio**

The point $P$ dividing the segment from $z_1$ to $z_2$ internally in the ratio $m : n$ is

$$P = \frac{mz_2 + nz_1}{m + n}.$$

The midpoint ($m = n = 1$) is $\dfrac{z_1 + z_2}{2}$. External division in the ratio $m : n$ is $P = \dfrac{mz_2 - nz_1}{m - n}$.

*Why.* $P$ lies on segment $z_1z_2$ with $\dfrac{|P - z_1|}{|z_2 - P|} = \dfrac{m}{n}$ and $P - z_1$ points the same direction as $z_2 - P$, i.e. $n(P - z_1) = m(z_2 - P)$. Solving for $P$ gives the stated formula. ■

**Example**

Find the point dividing the segment from $z_1 = 1 + 2i$ to $z_2 = 5 + 8i$ in the ratio $2 : 3$.

*Solution.* $P = \dfrac{2(5 + 8i) + 3(1 + 2i)}{5} = \dfrac{10 + 16i + 3 + 6i}{5} = \dfrac{13 + 22i}{5} = 2.6 + 4.4i.$

(Verified: $|P - z_1| : |z_2 - P| = 2.884\ldots : 4.327\ldots = 2 : 3$ exactly.)

**Centroid**

The centroid of a triangle with vertices $z_1, z_2, z_3$ is $\dfrac{z_1 + z_2 + z_3}{3}$ — the natural three-point extension of the midpoint formula.

## 3 Rotation: The Signature Move

This is the single most useful geometric tool complex numbers provide — one that coordinate geometry makes painful and complex numbers make automatic.

**Theorem Rotation About a Point**

To rotate the point $z$ by angle $\theta$ (counterclockwise) about the point $z_0$, the image is

$$z' = z_0 + (z - z_0)\, e^{i\theta}.$$

*Why.* $z - z_0$ is the vector from $z_0$ to $z$. Multiplying any complex number by $e^{i\theta}$ rotates it by $\theta$ about the *origin* (Part 2) while preserving its modulus. So $(z - z_0)e^{i\theta}$ is that same vector, rotated about the origin — and adding back $z_0$ re-anchors the rotation at $z_0$ instead of the origin.

**Example**

Rotate $z = 4 + 2i$ by $60^\circ$ about $z_0 = 1 + i$.

*Solution.* $z - z_0 = 3 + i$. $e^{i\pi/3} = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$.

$$(3+i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{3}{2} + i\frac{3\sqrt{3}}{2} + \frac{i}{2} - \frac{\sqrt{3}}{2} = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{3\sqrt{3}}{2} + \frac{1}{2}\right).$$

Then $z' = z_0 +$ this. (Numerically $z' \approx 1.634 + 4.098i$ — and indeed $|z' - z_0| = |z - z_0| = \sqrt{10}$, confirming the rotation preserved distance from the pivot.)

**Building an Equilateral Triangle**

If $z_1, z_2$ are two vertices of an equilateral triangle, the third vertex is obtained by rotating $z_2$ about $z_1$ by $\pm 60^\circ$:

$$z_3 = z_1 + (z_2 - z_1)e^{\pm i\pi/3}.$$

This single formula replaces an entire page of coordinate-geometry casework.

## 4 Collinearity and the Equilateral Triangle Condition

### 4.1 When Are Three Points Collinear?

**Theorem Collinearity Test**

Points $z_1, z_2, z_3$ are collinear if and only if

$$\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R},$$

equivalently $(z_3 - z_1)\overline{(z_2 - z_1)}$ is real, equivalently

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

*Why.* $z_3 - z_1$ and $z_2 - z_1$ are vectors from the common point $z_1$. Their ratio is real exactly when they point along the same line (same or opposite direction) — this is the collinearity condition itself, restated using the "same line through the origin" idea from Part 4's $\tan\theta = \tan\alpha$ discussion.

### 4.2 The Equilateral Triangle Condition

**Theorem Equilateral Triangle**

A triangle with vertices $z_1, z_2, z_3$ is equilateral if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

*Proof sketch.* Equilateral means rotating $z_2$ about $z_1$ by $60^\circ$ lands on $z_3$ *or* rotating $z_3$ about $z_1$ by $60^\circ$ lands on $z_2$ — i.e. $\dfrac{z_3 - z_1}{z_2 - z_1} = e^{\pm i\pi/3}$. Since $e^{i\pi/3}$ satisfies $\omega^2 - \omega + 1 = 0$ (a primitive 6th root of unity), substituting and clearing denominators through symmetric manipulation of all three vertices yields the symmetric identity above. (A fully symmetric equivalent worth remembering: $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$ for an equilateral triangle traversed consistently.)

**Example**

Verify that $z_1 = 0$, $z_2 = 1$, $z_3 = e^{i\pi/3}$ form an equilateral triangle.
*Solution.* $z_1^2 + z_2^2 + z_3^2 = 0 + 1 + e^{i2\pi/3}$ and $z_1z_2 + z_2z_3 + z_3z_1 = 0 + e^{i\pi/3} + 0 = e^{i\pi/3}$. Since $e^{i2\pi/3} = \cos 120^\circ + i\sin 120^\circ = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, the left side is $1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, which equals the right side exactly. ✓

## 5 Perpendicularity and Parallelism

**Theorem Perpendicularity Test**

Segments $z_1z_2$ and $z_3z_4$ are perpendicular if and only if $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary. They are parallel if and only if that ratio is real.

*Why.* The ratio of two vectors is real exactly when they share a direction (parallel case, extending the collinearity idea); it is purely imaginary exactly when one is a $90^\circ$-rotation of the other (multiplying by a purely imaginary number rotates by $\pm 90^\circ$, as noted already for $i$ itself in Part 2).

## 6 Concyclicity: Four Points on a Circle

**Theorem Concyclicity Test**

Four points $z_1, z_2, z_3, z_4$ (no three collinear) lie on a common circle or line if and only if the **cross-ratio**

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is real.

**Example**

Verify $1, i, -1, -i$ (the 4th roots of unity, from Part 3) are concyclic.
*Solution.* Cross-ratio $= \dfrac{(1-(-1))(i-(-i))}{(i-(-1))(1-(-i))} = \dfrac{2 \cdot 2i}{(1+i)(1+i)} = \dfrac{4i}{2i} = 2$, real. ✓(Expected — they lie on the unit circle by construction.)

## 7 Area of a Triangle

**Theorem Area from Complex Coordinates**

The area of the triangle with vertices $z_1, z_2, z_3$ is

$$\text{Area} = \left| \frac{1}{4i} \Big[ (z_1\bar{z}_2 - \bar{z}_1 z_2) + (z_2\bar{z}_3 - \bar{z}_2 z_3) + (z_3\bar{z}_1 - \bar{z}_3 z_1) \Big] \right|.$$

**Remark**

Each bracketed term $z_j\bar{z}_k - \bar{z}_j z_k = 2i\,\mathrm{Im}(z_j\bar{z}_k)$ is purely imaginary (Part 1, conjugate properties), so the whole sum divided by $4i$ is guaranteed real — the absolute value is only needed because the raw (unsigned) expression flips sign with the orientation (clockwise vs. counterclockwise) in which $z_1, z_2, z_3$ are listed, exactly like the shoelace formula in coordinate geometry, which this *is*, rewritten in complex form.

## 8 Loci in the Argand Plane

Part 1 previewed this with a perpendicular-bisector example. Here is the complete picture.

**Standard Loci**

- $|z - z_0| = r$ : circle, centre $z_0$, radius $r$.
- $|z - z_1| = |z - z_2|$ : perpendicular bisector of $z_1z_2$ (Part 1).
- $|z - z_1| + |z - z_2| = 2a$ (constant, $> |z_1 - z_2|$) : ellipse with foci $z_1, z_2$.
- $\big||z - z_1| - |z - z_2|\big| = 2a$ : hyperbola with foci $z_1, z_2$.
- $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ (constant): an arc of a circle through $z_1, z_2$ (the locus of points from which segment $z_1z_2$ subtends a fixed angle $\alpha$) — the complex-number form of the inscribed-angle theorem.
- $\dfrac{|z - z_1|}{|z - z_2|} = k$ (constant, $k \neq 1$): a circle, called the **Apollonius circle** of $z_1, z_2$ with ratio $k$ (when $k = 1$ this degenerates to the perpendicular bisector above).

**Apollonius Circle, Explicitly**

Find the locus $|z| = 2|z - 4|$.
*Solution.* Let $z = x + iy$. $x^2 + y^2 = 4\big[(x-4)^2 + y^2\big] \Rightarrow x^2 + y^2 = 4x^2 - 32x + 64 + 4y^2$

$$\Rightarrow 3x^2 + 3y^2 - 32x + 64 = 0 \Rightarrow x^2 + y^2 - \frac{32}{3}x + \frac{64}{3} = 0,$$

a circle centred at $\left(\dfrac{16}{3}, 0\right)$ with radius $\dfrac{8}{3}$ (verified numerically: a point on this circle indeed satisfies $|z|/|z - 4| = 2$).

## 9 Worked Examples: Full Problems

### JEE-style

If $z_1, z_2, z_3$ are the vertices of an equilateral triangle inscribed in $|z| = r$, and $z_1 + z_2 + z_3 = 0$, show this is automatically consistent with the equilateral-triangle condition of Section 4.2.
*Solution.* Since the centroid of an equilateral triangle coincides with its circumcentre, $z_1 + z_2 + z_3 = 0$ (centroid at origin, matching circumcentre at origin) is expected. Squaring: $z_1^2 + z_2^2 + z_3^2 = -2(z_1z_2 + z_2z_3 + z_3z_1)$. Combined with $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ (the equilateral condition), we get $3(z_1z_2 + z_2z_3 + z_3z_1) = 0$, so both symmetric sums vanish — consistent, and in fact this is precisely the condition satisfied by the vertices of *any* triangle inscribed in a circle whose centroid is at the centre and which is equilateral (e.g. the cube roots of unity from Part 3 satisfy exactly this).

### Olympiad-style

$ABCD$ is a square with vertices labelled counterclockwise, represented by complex numbers $a, b, c, d$. Express $c$ and $d$ in terms of $a, b$ alone.
*Solution.* Rotating vector $\overrightarrow{AB} = b - a$ by $90^\circ$ counterclockwise about $B$ gives $\overrightarrow{BC}$, so $c - b = i(b - a)$, giving $c = b + i(b - a)$. Similarly, translating that same rotated vector from $A$ gives $d = a + i(b - a)$. Both $c$ and $d$ are expressed purely in terms of $a, b$, illustrating how rotation collapses an entire square's worth of coordinates into two starting points.

### Practice Problems

**1.** Find the point dividing the segment from $2 - i$ to $6 + 7i$ externally in the ratio $3 : 1$.

**2.** Rotate $z = 2 + 3i$ by $90^\circ$ about the origin, then about $z_0 = 1 - i$. Compare the two results.

**3.** Prove that $z_1 = 2,\ z_2 = 2i,\ z_3 = -2$ do *not* form an equilateral triangle, using the Section 4.2 test.

**4.** Show that the points $1,\ i,\ -1,\ -i,\ \frac{1+i}{\sqrt{2}}$ are concyclic (five points on the unit circle) using any three plus the cross-ratio test on a fourth.

**5.** Find the area of the triangle with vertices $0,\ 3 + 4i,\ -2 + 6i$ using the complex-number area formula, and verify against the standard shoelace formula in $(x, y)$ coordinates.

**6.** Describe the locus $\left|\dfrac{z-2}{z+2}\right| = 1$, and identify it among the standard loci above.

**7.** If $z_1, z_2$ are fixed and $z$ moves so that $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \dfrac{\pi}{2}$ always, describe the locus of $z$ precisely (relate it to Thales' theorem).

**8.** (Olympiad) Prove that if $z_1, z_2, z_3, z_4$ are concyclic, then so are $\dfrac{1}{z_1 - a}, \dfrac{1}{z_2 - a}, \dfrac{1}{z_3 - a}, \dfrac{1}{z_4 - a}$ for any fixed $a$ not equal to any $z_j$ (i.e. inversion preserves concyclicity, using the cross-ratio test).

**9.** (Olympiad) $P$ is a point such that $|PA| = |PB| = |PC|$ for triangle $ABC$ with vertices $a, b, c$. Express $P$ in terms of $a, b, c$ using the perpendicular-bisector locus twice, and identify $P$.

**10.** Given a triangle with vertices $z_1, z_2, z_3$, find the complex number representing the point where the triangle's medians meet, and prove it using the section formula.

## What's Next

Part 6 extends this geometric toolkit toward **coordinate geometry and conics** — straight lines, circles, and conic sections expressed and manipulated entirely in complex-number form, including complex-number proofs of classical circle theorems. Part 7 then turns back to **algebra**: polynomial equations with complex roots, conjugate root pairs, and the structure of equations whose coefficients are themselves complex.

PART 6

# Coordinate Geometry & Conics in Complex Form

## 1 The General Equation of a Line

Part 5 showed that $z_1, z_2, z_3$ are collinear iff $\dfrac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}$. Turning that condition into an equation for *all* points $z$ on the line through two fixed points $z_1, z_2$ gives a standard, reusable template.

**Theorem Line Through Two Points**

The line through $z_1$ and $z_2$ consists of all $z$ satisfying

$$(\bar{z}_2 - \bar{z}_1)z - (z_2 - z_1)\bar{z} = \bar{z}_1 z_2 - z_1 \bar{z}_2.$$

*Why.* This is exactly $\dfrac{z - z_1}{z_2 - z_1} = \overline{\left(\dfrac{z - z_1}{z_2 - z_1}\right)}$ (a ratio equal to its own conjugate is real) cleared of denominators.

**Definition General Complex Form of a Line**

Every straight line in the Argand plane can be written as

$$\alpha z + \bar{\alpha}\bar{z} + \beta = 0, \qquad \alpha \in \mathbb{C} \setminus \{0\},\ \ \beta \in \mathbb{R}.$$

**Remark**

Writing $z = x + iy$ and $\alpha = p + iq$, the equation $\alpha z + \bar{\alpha}\bar{z} + \beta = 0$ expands to $2px - 2qy + \beta = 0$ — an ordinary real linear equation in $x, y$. This confirms the complex form really does describe a general line, and shows how to translate back to Cartesian form whenever needed.

## 2 The General Equation of a Circle

**Definition General Complex Form of a Circle**

Every circle in the Argand plane can be written as

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0, \qquad b \in \mathbb{C},\ c \in \mathbb{R},\ |b|^2 > c.$$

**Theorem Centre and Radius**

The circle $z\bar{z} + \bar{b}z + b\bar{z} + c = 0$ has centre $-b$ and radius $\sqrt{|b|^2 - c}$.

*Proof.* Complete the square: $(z+b)(\bar{z}+\bar{b}) = z\bar{z} + \bar{b}z + b\bar{z} + |b|^2$, so the given equation is equivalent to $|z+b|^2 = |b|^2 - c$, i.e. $|z-(-b)|^2 = |b|^2 - c$ — exactly the equation $|z - z_0| = r$ with $z_0 = -b$, $r = \sqrt{|b|^2 - c}$. ■

**Connection to Part 5**

Setting $c = |z_0|^2 - r^2$ and $b = -z_0$ recovers the familiar form $|z - z_0| = r$ used throughout Part 5's loci discussion; the general form here is simply that equation multiplied out, which is often the more convenient shape for algebraic manipulation (e.g. finding intersections with lines).

**Example**

Find the centre and radius of the circle $z\bar{z} - (2-3i)z - (2+3i)\bar{z} - 10 = 0$.
*Solution.* Here $\bar{b} = -(2-3i) \Rightarrow b = -(2+3i)$, and $c = -10$. Centre $= -b = 2+3i$. Radius $= \sqrt{|-(2+3i)|^2 - (-10)} = \sqrt{13+10} = \sqrt{23}$.

## 3 Tangents to a Circle

**Theorem Tangent at a Point on the Circle**

The tangent to the circle $|z| = r$ at the point $z_1$ (with $|z_1| = r$) on it is

$$z\bar{z}_1 + \bar{z}z_1 = 2r^2, \qquad \text{equivalently } \operatorname{Re}(z\bar{z}_1) = r^2.$$

For the circle $|z - z_0| = r$, translate: the tangent at $z_1$ on the circle is

$$(z - z_0)\overline{(z_1 - z_0)} + \overline{(z - z_0)}(z_1 - z_0) = 2r^2.$$

*Why.* The tangent at $z_1$ is perpendicular to the radius $z_1$ (for the origin-centred case). By Part 5's perpendicularity test, points $z$ on this tangent satisfy $\dfrac{z - z_1}{z_1}$ purely imaginary, i.e. it equals the negative of its own conjugate: $\dfrac{z - z_1}{z_1} = -\dfrac{\bar{z} - \bar{z}_1}{\bar{z}_1}$. Clearing denominators and simplifying using $z_1\bar{z}_1 = r^2$ gives the stated equation.

**Example**

Find the tangent to $|z| = 5$ at $z_1 = 3 + 4i$ (check: $|z_1| = 5$ ✓), and verify a point on it.
*Solution.* Tangent: $z(3-4i) + \bar{z}(3+4i) = 50$. Writing $z = x + iy$: $(3-4i)(x+iy) + (3+4i)(x-iy) = 6x + 8y = 50$, i.e. $3x + 4y = 25$ — the familiar Cartesian tangent-at-a-point formula for a circle centred at the origin, reproduced instantly.

**Theorem Tangency Condition: Line Touches Circle**

The line $\alpha z + \bar{\alpha}\bar{z} + \beta = 0$ is tangent to the circle $|z - z_0| = r$ if and only if

$$\frac{|\alpha z_0 + \bar{\alpha}\bar{z}_0 + \beta|}{2|\alpha|} = r,$$

i.e. the perpendicular distance from the centre to the line equals the radius.

## 4 Reflection of a Point in a Line

**Theorem Reflection Formula**

The reflection of a point $z$ in the line through $z_1, z_2$ is

$$z' = z_1 + \overline{\left(\frac{z - z_1}{z_2 - z_1}\right)}(z_2 - z_1).$$

*Why.* Reflection preserves the foot of perpendicular and flips the perpendicular component of the vector $z - z_1$ (relative to the line's direction) while preserving the parallel component — exactly the effect of conjugating the ratio $\frac{z - z_1}{z_2 - z_1}$ (which measures position relative to the line's own direction) before converting back.

**Example**

Reflect $z = 3 + i$ in the line through 0 and $1 + i$ (the line $y = x$).

*Solution.* $z_1 = 0$, $z_2 = 1 + i$. $\frac{z - z_1}{z_2 - z_1} = \frac{3 + i}{1 + i} = \frac{(3 + i)(1 - i)}{2} = \frac{4 - 2i}{2} = 2 - i$. Conjugate: $2 + i$. So $z' = 0 + (2 + i)(1 + i) = 2 + 2i + i - 1 = 1 + 3i$ — exactly $(1, 3)$, the mirror image of $(3, 1)$ across $y = x$, as expected by inspection.

## 5 Conics: Complex Parametrization

Part 5 gave loci definitions for the ellipse and hyperbola via sums/differences of distances. Here they get explicit parametric forms, extending the circle's $z = re^{i\theta}$ parametrization from Part 3.

**Ellipse**

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is parametrized by

$$z(\theta) = a\cos\theta + ib\sin\theta, \qquad \theta \in [0, 2\pi).$$

This is a direct generalization of the circle's $z = re^{i\theta} = r\cos\theta + ir\sin\theta$ (the case $a = b = r$), with the two axes independently rescaled.

**Example**

Verify $z(\theta) = 5\cos\theta + 3i\sin\theta$ traces the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

*Solution.* $x = 5\cos\theta$, $y = 3\sin\theta \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = \cos^2\theta + \sin^2\theta = 1$ for every $\theta$. ✓

**Foci, in This Language**

For the ellipse above with $a > b$, the foci are at $z = \pm c$ where $c = \sqrt{a^2 - b^2}$ (both real, sitting on the real axis in this standard orientation) — matching Part 5's definition of an ellipse as the locus with $|z - z_1| + |z - z_2| = 2a$ for foci $z_1 = -c$, $z_2 = c$.

**Hyperbola**

The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is parametrized using cosh, sinh instead of cos, sin:

$$z(t) = a\cosh t + ib\sinh t, \qquad t \in \mathbb{R},$$

one branch at a time (the identity $\cosh^2 t - \sinh^2 t = 1$ plays the role that $\cos^2\theta + \sin^2\theta = 1$ played for the ellipse). This is a reminder that $\cosh t = \cos(it)$ and $\sinh t = -i\sin(it)$ — the hyperbolic functions are literally the circular functions evaluated at imaginary arguments, one more instance of Euler's formula connecting two seemingly separate topics.

## 6 Worked Examples

**JEE-style**

Find the equation of the circle passing through the points $1, i, -1$ (three points determine a circle).

*Solution.* All three points have modulus 1, so $|z| = 1$ (i.e. $z\bar{z} = 1$, matching the general form with $b = 0, c = -1$) already passes through all three — no further work needed once the pattern is spotted, though in general one would substitute all three points into $z\bar{z} + \bar{b}z + b\bar{z} + c = 0$ and solve the resulting linear system for $b, c$.

**Olympiad-style**

$P$ is a point on the ellipse $z(\theta) = a\cos\theta + ib\sin\theta$. Show the tangent to the ellipse at $P$ makes intercepts on the axes whose product depends only on $\theta$ in a specific way, by first finding the Cartesian tangent-line equation at $P = (a\cos\theta, b\sin\theta)$.

*Solution sketch.* The standard tangent to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$, with $x$-intercept $\dfrac{a}{\cos\theta}$ and $y$-intercept $\dfrac{b}{\sin\theta}$; their product is $\dfrac{ab}{\sin\theta\cos\theta} = \dfrac{2ab}{\sin 2\theta}$ — reusing Part 2's double-angle identity to compress the answer, illustrating how the trigonometric machinery built earlier in this book keeps resurfacing in later, seemingly unrelated topics.

**Practice Problems**

1. Find the general complex-form equation of the line through $2 - i$ and $4 + 3i$, then convert it to Cartesian form.
2. Find the centre and radius of the circle $z\bar{z} + (1 + 2i)z + (1 - 2i)\bar{z} - 4 = 0$.
3. Find the tangent to the circle $|z| = 13$ at the point $5 + 12i$, and convert to Cartesian form.
4. Determine whether the line $x + y = 6$ is tangent to, secant to, or misses the circle $|z - (1 + 1i)| = 3$, using the tangency-distance formula.
5. Reflect the point $z = 4 + 2i$ in the line through 1 and $i$.
6. Write the parametric form of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$ using complex numbers, and find the two points where $\theta = \pi/4$.
7. Find the foci of the ellipse traced by $z(\theta) = 4\cos\theta + 3i\sin\theta$.

**8.** Show that the circle $|z-2|=3$ and the circle $|z+1|=2$ do not intersect (find the distance between centres and compare with the sum/difference of radii).

**9.** (Olympiad) Prove that the general equation $z\bar{z}+\bar{b}z+b\bar{z}+c=0$ represents a single point (a "point circle") when $|b|^2=c$, and represents no points at all (an empty locus) when $|b|^2<c$.

**10.** (Olympiad) $A, B, C$ are points on the circle $|z|=r$. Using the tangent formula, show that the tangents at $A$ and $B$ meet at the point $z=\dfrac{2ab}{a+b}$ where $a, b$ are the complex numbers for $A, B$ (assume $A \neq B$, neither diametrically opposite).

## What's Next

Part 7 turns fully back to **algebra**: polynomial equations with complex coefficients and complex roots, the conjugate-root-pairs theorem for real polynomials, and how the geometric pictures built in Parts 3, 5, and 6 illuminate where the roots of a polynomial equation must lie.

PART 7

# Equations with Complex Roots

## 1 The Fundamental Theorem of Algebra

**Fundamental Theorem of Algebra**

Every polynomial equation of degree $n \geq 1$ with complex coefficients has exactly $n$ roots in $\mathbb{C}$, counted with multiplicity.

**Remark**

This is the reason Part 3's $z^n = 1$ had exactly $n$ solutions, and it is why $\mathbb{C}$ is called **algebraically closed** — unlike $\mathbb{R}$, where $x^2 + 1 = 0$ has no solution, *every* polynomial equation is solvable in $\mathbb{C}$, no exceptions. A full proof needs tools beyond this book (complex analysis or topology), but everything in this chapter assumes and uses this guarantee.

## 2 Complex Square Roots: A Prerequisite

Before solving quadratics with complex coefficients, we need $\sqrt{z}$ for an arbitrary complex $z$.

**Theorem Square Root Formula**

For $z = x + iy \neq 0$ with $r = |z|$, the two square roots of $z$ are

$$\pm\left(\sqrt{\frac{r+x}{2}} + i\,\operatorname{sgn}(y)\sqrt{\frac{r-x}{2}}\right),$$

where $\operatorname{sgn}(y) = 1$ if $y \geq 0$ and $-1$ if $y < 0$.

*Why.* Seek $w = a + ib$ with $w^2 = z$: $a^2 - b^2 = x$ and $2ab = y$. Also $|w|^2 = |z| \Rightarrow a^2 + b^2 = r$. Adding/subtracting with $a^2 - b^2 = x$ gives $a^2 = \frac{r+x}{2}$, $b^2 = \frac{r-x}{2}$; the sign of $ab$ (matching sign of $y$) fixes the relative sign of $a, b$. ■

**Remark**

An equivalent (often faster) route: write $z = re^{i\phi}$ and take $\sqrt{z} = \sqrt{r}\,e^{i\phi/2}$, a direct application of Part 3's $n$th-root machinery with $n = 2$. Both routes agree; use whichever is more

convenient for the numbers at hand.

**Example**

Find $\sqrt{5+12i}$.
*Solution.* $r = \sqrt{25+144} = 13$. $a = \sqrt{(13+5)/2} = 3$, $b = \sqrt{(13-5)/2} = 2$, and $y = 12 > 0$ so $b$ is positive. $\sqrt{5+12i} = \pm(3+2i)$. *Check:* $(3+2i)^2 = 9+12i-4 = 5+12i$. ✓

## 3 Quadratic Equations with Complex Coefficients

**Theorem General Quadratic Formula**

The equation $az^2+bz+c=0$ ($a \neq 0$, $a, b, c \in \mathbb{C}$) has roots

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a},$$

using the complex square root of Section 2. The two roots coincide (a repeated root) exactly when $b^2-4ac=0$.

The derivation (completing the square) is identical to the real case — nothing about it used $b^2-4ac \geq 0$; that condition was only ever needed to keep the root *real*, which is irrelevant once we allow complex answers.

**Example**

Solve $z^2-(3+i)z+(4+3i)=0$.
*Solution.* $b^2-4ac = (3+i)^2-4(4+3i) = (9+6i-1)-(16+12i) = (8+6i)-(16+12i) = -8-6i$. Using Section 2's method: $r = \sqrt{64+36} = 10$, $a = \sqrt{(10-8)/2} = 1$, $b = \sqrt{(10+8)/2} = 3$; since $\mathrm{Im}(-8-6i) = -6 < 0$, take $b$ negative: $\sqrt{-8-6i} = \pm(1-3i)$.

$$z = \frac{(3+i) \pm (1-3i)}{2} \implies z = 2-i \text{ or } z = 1+2i.$$

*Verification:* sum of roots $= 3+i$ ✓(matches $-(-(3+i)) = 3+i$); product $= (2-i)(1+2i) = 2+4i-i+2 = 4+3i$ ✓.

**The Discriminant No Longer Signals "Real vs. Non-Real"**

For a *real*-coefficient quadratic, $b^2-4ac<0$ tells you the roots are non-real. For a genuinely *complex*-coefficient quadratic, "$b^2-4ac$ negative" doesn't even make sense as a comparison (complex numbers aren't ordered) — the discriminant only tells you whether the two roots coincide ($=0$) or are distinct ($\neq 0$).

## 4 Real-Coefficient Polynomials: The Conjugate Root Theorem

This is the single most important structural fact about polynomial equations that students actually meet in real-coefficient settings — factorization, root-finding, degree-counting arguments all lean on it.

**Theorem Conjugate Root Theorem**

If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has **all real coefficients** and $z_0$ is a root, then $\bar{z}_0$ is also a root.

*Proof.* Since $p(z_0) = 0$, take the conjugate of both sides:

$$\overline{p(z_0)} = \overline{a_n z_0^n + \cdots + a_0} = a_n \bar{z}_0^n + \cdots + a_0 = p(\bar{z}_0),$$

using that each $a_k$ is real (so $\bar{a}_k = a_k$) and that conjugation distributes over sums and products (Part 1). So $p(\bar{z}_0) = \overline{p(z_0)} = \bar{0} = 0$. ■

**Immediate Consequences**

- Non-real roots of a real polynomial always come in conjugate **pairs** $z_0, \bar{z}_0$ — never alone.
- A real polynomial of **odd** degree must have at least one **real** root (the non-real roots use up an even number of the $n$ total roots, leaving at least one unpaired root, which must be real).
- Each conjugate pair $z_0, \bar{z}_0$ contributes a real **quadratic factor** $(z - z_0)(z - \bar{z}_0) = z^2 - 2\,\mathrm{Re}(z_0)z + |z_0|^2$ to the factorization — this is how every real polynomial factors into real linear and real irreducible quadratic pieces, never needing a lone complex linear factor.

**Example**

A real-coefficient cubic has $2 + i$ as a root and leading coefficient 1. Find the cubic, given that its third root is real and equal to $-3$.
*Solution.* By the theorem, $2 - i$ is also a root. The cubic is

$$(z - (2+i))(z - (2-i))(z+3) = \big(z^2 - 4z + 5\big)(z+3) = z^3 - z^2 - 7z + 15.$$

(Verified real throughout, as required.)

## 5 Vieta's Formulas, Extended to Complex Roots

**Theorem Vieta's Formulas**

If $z_1, \ldots, z_n$ are the roots (with multiplicity) of $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$, then

$$\sum z_i = -\frac{a_{n-1}}{a_n}, \qquad \sum_{i<j} z_i z_j = \frac{a_{n-2}}{a_n}, \qquad \ldots, \qquad \prod z_i = (-1)^n \frac{a_0}{a_n}.$$

These hold identically whether the roots are real or complex — nothing in their derivation (matching coefficients in $a_n \prod(z - z_i) = a_n z^n + \cdots + a_0$) uses realness. This is often the fastest route to symmetric-function questions about roots, entirely avoiding the need to solve for the roots individually.

**Example**

If $\alpha, \beta, \gamma$ are the roots of $z^3 - 6z^2 + 11z - 6 = 0$, find $\alpha^2 + \beta^2 + \gamma^2$ without solving the cubic.
*Solution.* $\sum \alpha = 6$, $\sum \alpha\beta = 11$. Then $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta = 36 - 22 = 14$. (The roots here happen to be $1, 2, 3$ — real — but the method works unchanged even when some roots are

non-real.)

# 6 Equations Reducible by Substitution

## 6.1 Reciprocal Equations

**The $z + \frac{1}{z}$ Substitution**

An equation is called **reciprocal** if its coefficients read the same forwards and backwards (e.g. $z^4 + z^3 + z^2 + z + 1 = 0$, whose coefficients $1, 1, 1, 1, 1$ are palindromic). Dividing by $z^{n/2}$ (for even degree $n$) and grouping $z^k + z^{-k}$ terms lets the whole equation be rewritten in terms of $w = z + \dfrac{1}{z}$, reducing the degree by half.

**Example**

Solve $z^4 + z^3 + z^2 + z + 1 = 0$.

*Solution.* This is exactly $\dfrac{z^5 - 1}{z - 1} = 0$ for $z \neq 1$ (Part 3's factorization of $z^n - 1$), so its roots are the four *non-trivial* 5th roots of unity: $z_k = \operatorname{cis}\left(\dfrac{2k\pi}{5}\right)$ for $k = 1, 2, 3, 4$. This is the cleanest possible illustration of why Part 3's roots-of-unity toolkit matters for solving equations directly, without needing the reciprocal substitution machinery at all — though dividing by $z^2$ and setting $w = z + 1/z$ (giving $w^2 + w - 1 = 0$, solvable by the ordinary quadratic formula) is the standard fallback method when an equation isn't recognizable as a roots-of-unity factor.

# 7 Where Do the Roots Live? Bounding Root Modulus

For Olympiad-style problems it is often enough to know *roughly* where all roots of a polynomial lie, without solving anything.

**Theorem Cauchy's Bound**

Every root $z$ of $z^n + a_{n-1}z^{n-1} + \cdots + a_0 = 0$ (monic, i.e. leading coefficient 1) satisfies

$$|z| \leq 1 + \max_{0 \leq k \leq n-1} |a_k|.$$

*Sketch.* If $|z| > 1$, then from the equation $z^n = -(a_{n-1}z^{n-1} + \cdots + a_0)$, the triangle inequality (Part 1) gives $|z|^n \leq M(|z|^{n-1} + \cdots + |z| + 1) = M \cdot \dfrac{|z|^n - 1}{|z| - 1}$ where $M = \max |a_k|$; simplifying this inequality forces $|z| \leq 1 + M$. ■

**Example**

Bound the roots of $z^3 + 3z^2 - 5z + 2 = 0$ without solving it.

*Solution.* $M = \max(3, 5, 2) = 5$, so every root satisfies $|z| \leq 6$. (The actual roots are approximately $-4.28$ and $0.64 \pm 0.24i$ — comfortably inside the bound, confirming it is a valid, if not tight, estimate.)

# 8 Worked Examples

**JEE-style**

If $1, \omega, \omega^2$ are the cube roots of unity, find the equation whose roots are $1+\omega$, $1+\omega^2$, 2.
*Solution.* Recall $1+\omega+\omega^2 = 0$ so $1+\omega = -\omega^2$ and $1+\omega^2 = -\omega$. Sum of the three roots $= -\omega^2 - \omega + 2 = -(\omega+\omega^2)+2 = -(-1)+2 = 3$. Sum of products in pairs: $(-\omega^2)(-\omega)+(-\omega)(2)+(2)(-\omega^2) = \omega^3 - 2\omega - 2\omega^2 = 1 - 2(\omega+\omega^2) = 1+2 = 3$. Product $= (-\omega^2)(-\omega)(2) = 2\omega^3 = 2$. By Vieta's, the cubic is $z^3 - 3z^2 + 3z - 2 = 0$.

**Olympiad-style**

Show that if $p(z)$ is a real-coefficient polynomial of degree $n$ with $p(i) = 0$, then $n \geq 2$ and $(z^2+1)$ divides $p(z)$.
*Solution.* By the Conjugate Root Theorem, $\bar{i} = -i$ is also a root. Since $i \neq -i$, these are two distinct roots, forcing $n \geq 2$. The minimal real polynomial vanishing at both $i$ and $-i$ is $(z-i)(z+i) = z^2+1$; since $p$ vanishes at both roots of this quadratic, polynomial division confirms $z^2+1 \mid p(z)$.

**Practice Problems**

1. Find both square roots of $-15+8i$.
2. Solve $z^2 - (2+3i)z + (-1+3i) = 0$.
3. A real-coefficient quartic has roots $1+i$ and $2-i$ (each with the required conjugate partner). Find the quartic.
4. If $\alpha, \beta, \gamma, \delta$ are the roots of $z^4 - 2z^3 + 3z^2 - 4z + 5 = 0$, find $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$ using Vieta's formulas only.
5. Solve $2z^4 + 3z^3 - 16z^2 + 3z + 2 = 0$, a reciprocal equation, using the $w = z + 1/z$ substitution.
6. Prove that a real-coefficient quintic (degree 5) polynomial must have at least one real root, generalizing the odd-degree argument in Section 4.
7. Use Cauchy's bound to show every root of $z^5 - z + 16 = 0$ satisfies $|z| \leq 17$, and comment on whether this bound looks tight.
8. If $\omega$ is a non-real cube root of unity, find the quadratic equation satisfied by $\omega$ alone (i.e. the minimal polynomial of $\omega$), using $1+\omega+\omega^2 = 0$.
9. (Olympiad) Prove that if all roots of a monic real polynomial $p(z)$ have $|z| = 1$ (all roots on the unit circle), then the coefficients of $p$ are palindromic (i.e. $p$ is a reciprocal polynomial as in Section 6.1).
10. (Olympiad) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be monic with real coefficients and all roots real. Prove that between any two real roots of $p$, there lies at least one real root of $p'(z)$ (Rolle's theorem), and explain why this is a statement purely about the *real* roots, unaffected by anything in this chapter about non-real roots.

## What's Next

Part 8, the final part, is devoted entirely to **Olympiad technique**: a deeper toolkit built from everything so far — inequalities involving complex numbers (triangle inequality sharpened, and complex Cauchy-Schwarz), functional-equation-style problems, and a curated set of hard, fully-worked Olympiad problems that draw on multiple parts of this book at once, tying the whole subject together.

PART 8

# Olympiad Techniques — The Complete Toolkit in Action

## 1 Sharpening the Triangle Inequality

Part 1 stated $|z_1 + z_2| \leq |z_1| + |z_2|$. Olympiad problems usually need the **equality condition** as much as the inequality itself.

**Theorem Equality in the Triangle Inequality**

$|z_1 + z_2| = |z_1| + |z_2|$ if and only if $z_1$ and $z_2$ point in the same direction — i.e. one is a non-negative real multiple of the other ($z_1 = tz_2$ for some $t \geq 0$, or one of them is 0).

*Why.* Geometrically, the "triangle" formed by $0, z_1, z_1 + z_2$ degenerates to a straight segment exactly when $z_1, z_2$ are collinear with 0 and on the same side — matching Part 5's collinearity criterion, restricted to the same-direction case.

**Theorem Generalized ($n$-term) Triangle Inequality**

For any complex numbers $z_1, \ldots, z_n$,

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|,$$

with equality iff all the $z_k$ (that are non-zero) point in the same direction.

**Example**

If $|z_1| = |z_2| = \cdots = |z_n| = 1$ and $z_1 + z_2 + \cdots + z_n = n$, show all $z_k = 1$.
*Solution.* $n = |z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n| = n$, so equality holds in the generalized triangle inequality. Hence all $z_k$ point the same direction; combined with $|z_k| = 1$, all $z_k$ equal the same unit complex number $u$. Then $nu = n \Rightarrow u = 1$, so every $z_k = 1$.

## 2 The Complex Cauchy–Schwarz Inequality

**Theorem Cauchy–Schwarz for Complex Numbers**

For complex numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\left|\sum_{k=1}^{n} a_k \bar{b}_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right),$$

with equality iff $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional.

**Application**

If $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = n$, show $|z_1 + z_2 + \cdots + z_n| \leq n$.
*Solution.* Take $a_k = z_k$, $b_k = 1$ in Cauchy–Schwarz: $|\sum z_k|^2 \leq (\sum |z_k|^2)(n) = n \cdot n = n^2$, so $|\sum z_k| \leq n$.

## 3 The Complex-Substitution Trick

Many real-variable Olympiad problems (especially trigonometric ones) become tractable the moment a real quantity is recognized as $\mathrm{Re}(z)$ or $2\,\mathrm{Re}(z)$ for a well-chosen complex $z$ — the technique used repeatedly since Part 2.

**The Master Pattern**

Whenever a problem involves $\cos k\theta$, $\sin k\theta$, or sums of these, try setting $z = e^{i\theta}$ (or $z_k = e^{ik\theta}$) and rewriting the entire problem as a statement about $z$ and $|z| = 1$. Algebraic identities about $z$ (factoring, bounding $|z^n \pm 1|$, etc.) then translate back into trigonometric conclusions — often far more easily than direct trigonometric manipulation would allow.

**Example**

Find the maximum value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ given $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\theta_i > 0$.
*Solution (sketch, using convexity rather than direct complex substitution here for contrast).* This particular problem is cleanest by Jensen's inequality (concavity of cos on $(0, \pi)$), giving maximum $\dfrac{3}{2}$ at $\theta_1 = \theta_2 = \theta_3 = \pi/3$. It is included deliberately to flag a limit of the toolkit: **not every trigonometric optimization needs complex numbers** — recognizing when a direct real-variable technique (calculus, Jensen, AM-GM) is faster is itself part of Olympiad skill.

## 4 Extremal Modulus Problems

**Example**

Given $|z| = 2$, find the maximum and minimum values of $|z + 3 + 4i|$.
*Solution.* Let $w_0 = -(3 + 4i)$, $|w_0| = 5$. As $z$ ranges over $|z| = 2$, the point $z$ traces a circle of radius 2 centred at the origin; $|z - w_0|$ is the distance from $z$ to the fixed point $w_0$. This distance ranges between $|w_0| - 2 = 3$ (when $z$ is on the segment from 0 toward $w_0$, closest point) and $|w_0| + 2 = 7$ (farthest point, diametrically opposite). So min $= 3$, max $= 7$.

**The General Pattern**

For $|z - z_0| = r$ fixed and any fixed point $w$, $|z - w|$ ranges over $\big[\,|z_0 - w| - r,\ |z_0 - w| + r\,\big]$ (assuming $r \le |z_0 - w|$; otherwise the minimum is 0, since the fixed point lies inside or on the circle). This is a direct consequence of the triangle inequality applied to $z - w = (z - z_0) + (z_0 - w)$.

## 5 Curated Problems: The Toolkit, Fully Assembled

**Problem 1 — Algebra + Geometry, from Part 5**

If $a, b, c$ are complex numbers with $|a| = |b| = |c| = r > 0$ and $a + b + c = 0$, prove $a, b, c$ form an equilateral triangle inscribed in $|z| = r$.

*Solution.* Since $|a| = |b| = |c| = r$, all three lie on $|z| = r$ (inscribed, automatically). From $a + b + c = 0$: $c = -(a + b)$, so $|c|^2 = |a + b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}(a\bar{b}) = 2r^2 + 2\,\mathrm{Re}(a\bar{b})$. Since $|c| = r$, this gives $\mathrm{Re}(a\bar{b}) = -r^2/2$. Now compute $|a - b|^2 = |a|^2 + |b|^2 - 2\,\mathrm{Re}(a\bar{b}) = 2r^2 + r^2 = 3r^2$, so $|a - b| = r\sqrt{3}$. By the symmetry of the hypothesis in $a, b, c$, the same argument gives $|b - c| = |c - a| = r\sqrt{3}$ as well — all three sides equal, so the triangle is equilateral. ■ (This closes the loop on the very first Olympiad-style practice problem posed all the way back in Part 1.)

**Problem 2 — Roots of Unity + Inequalities**

If $\omega$ is a primitive $n$th root of unity ($n \ge 2$), prove $\displaystyle\sum_{k=1}^{n-1} \frac{1}{|1 - \omega^k|^2} = \frac{n^2 - 1}{12}$.

*Solution sketch.* Start from Part 3's identity $\prod_{k=1}^{n-1}(1 - \omega^k) = n$; take logarithmic derivatives of $\displaystyle\prod_{k=1}^{n-1}(z - \omega^k) = 1 + z + \cdots + z^{n-1}$ with respect to $z$ and evaluate cleverly at $z = 1$ (a standard but slightly advanced manipulation) to relate the sum of $\dfrac{1}{1 - \omega^k}$-type expressions to the coefficients of the polynomial; converting real parts of the resulting sum and using $|1 - \omega^k|^2 = (1 - \omega^k)(1 - \bar{\omega}^k)$ produces the stated closed form after simplification. This problem is flagged as a *template* for how far the roots-of-unity machinery from Part 3 can be pushed — full algebraic detail is a rewarding independent exercise once Parts 2–3 are internalized.

**Problem 3 — Geometry, from Part 5 + Part 6**

$ABC$ is a triangle with vertices $a, b, c$ on the circle $|z| = r$. $H$ is a point such that $H = a + b + c$. Prove $H$ is the orthocentre of $\triangle ABC$.

*Solution.* It suffices to show $\overrightarrow{AH} \perp \overrightarrow{BC}$ (and by symmetry the same holds for the other two vertex–side pairs). $H - a = b + c$. We must show $\dfrac{b + c}{c - b}$ is purely imaginary (Part 5's perpendicularity test). Since $|b| = |c| = r$, write $\bar{b} = r^2/b$, $\bar{c} = r^2/c$ (from $b\bar{b} = r^2$). Then

$$\overline{\left(\frac{b + c}{c - b}\right)} = \frac{\bar{b} + \bar{c}}{\bar{c} - \bar{b}} = \frac{r^2/b + r^2/c}{r^2/c - r^2/b} = \frac{c + b}{b - c} = -\frac{b + c}{c - b},$$

i.e. $\dfrac{b + c}{c - b}$ equals the negative of its own conjugate — exactly the purely-imaginary condition. ■ This is the standard, celebrated complex-number proof that the orthocentre of a triangle inscribed in a circle centred at the origin is simply the sum of its vertices — a fact that is

genuinely painful to prove with pure coordinate geometry.

**Problem 4 — Inequality, from Section 1–2**

For complex numbers $z_1, \ldots, z_n$ with $|z_k| \leq 1$ for all $k$, and real numbers $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum \lambda_k = 1$, prove $|\sum \lambda_k z_k| \leq 1$.
*Solution.* By the generalized triangle inequality, $|\sum \lambda_k z_k| \leq \sum \lambda_k |z_k| \leq \sum \lambda_k \cdot 1 = 1$ (using $\lambda_k \geq 0$ to pull it out of the modulus, and $|z_k| \leq 1$). ■ (This says: a weighted average of points inside the unit disc stays inside the unit disc — the disc is *convex*, proved here in one line using tools from Section 1.)

**Problem 5 — Trigonometric Sum via Roots of Unity**

Evaluate $\displaystyle\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

*Solution sketch.* From Part 3, $\displaystyle\prod_{k=1}^{n-1}(1-\omega^k) = n$ where $\omega = e^{2\pi i/n}$. Write $1-\omega^k = 1 - e^{2\pi ik/n} = -2i \sin\left(\frac{k\pi}{n}\right) e^{ik\pi/n}$ (factor out $e^{ik\pi/n}$ and use $e^{-i\theta} - e^{i\theta} = -2i\sin\theta$ appropriately). Taking the modulus of both sides of the product identity, the exponential factors have modulus 1 and drop out, leaving

$$\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n} = n \implies \prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

## 6 Closing Practice Set: Mixed, Cumulative

**Practice Problems**

1. Prove that $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ with equality iff $z_1, z_2$ point in the same direction (relate to Section 1).
2. If $|z| = 1$ and $z \neq \pm 1$, prove $\dfrac{z}{1+z^2}$ is real. (Combine Part 1's realness criterion with $\bar{z} = 1/z$.)
3. Given $|z-1| = 1$, find the maximum value of $|z|$ (Section 4's extremal-modulus pattern).
4. If $z_1, z_2, z_3$ are vertices of an equilateral triangle and $O$ is its centroid, prove $|z_1 - O| = |z_2 - O| = |z_3 - O|$ using the rotation formula from Part 5.
5. Use the complex-substitution trick to re-derive $\cos 5\theta$ in terms of $\cos\theta$ (revisit Part 2's method) and check your formula numerically at $\theta = 1$ radian.
6. Prove the parallelogram law $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ using Cauchy–Schwarz-style expansion (Part 1 revisited, now via Section 2's tools).
7. (Olympiad) If $|z| \leq 1$, prove $|z^n - 1| \leq n|z-1|$ for every positive integer $n$. (Hint: factor $z^n - 1$ and apply the generalized triangle inequality.)
8. (Olympiad) Find all complex numbers $z$ with $|z| = 1$ satisfying $z^6 + z^3 + 1 = 0$ (combine Part 7's equation-solving with Part 3's roots of unity).

**9.** (Olympiad) $P(z)$ is a monic polynomial of degree $n$ with all roots on $|z| = 1$. Prove $|P(0)| = 1$, and use this together with Part 7's Conjugate Root Theorem to describe the possible values of $P(0)$ when $P$ has real coefficients.

**10.** (Capstone) Using any combination of tools from this entire book, prove: if $z_1, z_2, z_3, z_4$ are the vertices (in order) of a cyclic quadrilateral inscribed in $|z| = r$, then $|z_1 - z_3| \cdot |z_2 - z_4| \leq |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_4 - z_1|$ (Ptolemy's inequality, with equality exactly when the quadrilateral is cyclic in the stated order — which it already is by hypothesis, so equality should hold; identify which theorem from Part 5 or 6 confirms this).

## The Book, Complete

Eight parts, one continuous thread: an algebraic definition in Part 1 became a point in a plane; that point's rotation under multiplication (Part 2) generated every trigonometric identity worth knowing; the special case of rotation by $\frac{2\pi}{n}$ (Part 3) produced regular polygons and a filter for extracting series terms; the same rotation, run backwards, solved every standard trigonometric equation and pinned down periods rigorously (Part 4); turned toward pure geometry, rotation and modulus proved classical Euclidean theorems without coordinates (Part 5) and rebuilt coordinate geometry and conics from scratch (Part 6); the algebra of roots closed the loop with the Conjugate Root Theorem and Vieta's formulas (Part 7); and every one of those tools converged in Part 8's Olympiad problems, several of which reached back explicitly to reuse results from Parts 1, 2, 3, and 5.

That convergence was the point from the very first page: complex numbers were never a separate topic sitting beside algebra, trigonometry, and geometry. They are the language in which those three turn out to be one subject.

# Index

Area from Complex Coordinates, 27
Argand Diagram, 3
Argument, 5

Building an Equilateral Triangle, 25

Cauchy's Bound, 38
Cauchy–Schwarz for Complex Numbers, 42
Centre and Radius, 31
Centroid, 24
Closure Under Multiplication, 15
Collinearity Test, 25
Complex Conjugate, 2
Concyclicity Test, 26
Conjugate Root Theorem, 37

De Moivre's Theorem, 10
Dividing a Segment in a Given Ratio, 24

Ellipse, 32
Equality in the Triangle Inequality, 41
Equilateral Triangle, 26
Euler's Formula, 9
Euler's Identity, 10

Fundamental Theorem of Algebra, 35

General Complex Form of a Circle, 30
General Complex Form of a Line, 30
General nth Roots, 16
General Quadratic Formula, 36
General Solution, 19, 20
Generalized (n-term) Triangle Inequality, 41
Geometric Meaning of the Sum, 15

Hyperbola, 33

Immediate Consequences, 37

Line Through Two Points, 30

Moduli Multiply, Arguments Add, 9
Modulus, 4

Perpendicularity Test, 26
Polar Form, 8
Primitive nth Root of Unity, 16
Product Formula, 8
Properties of Modulus, 4
Properties of the Conjugate, 3

Reducing Any Power of i, 2
Reflection Formula, 32
Rotation About a Point, 25

Square Root Formula, 35
Standard Loci, 27
Sum and Product of the nth Roots of Unity, 15

Tangency Condition: Line Touches Circle, 31
Tangent at a Point on the Circle, 31
The Filter, 17
The General Pattern, 11, 43
The Imaginary Unit, 1
The Master Pattern, 42
The Method, 12
The nth Roots of Unity, 14
The Period-by-GCD Rule, 21
The Unifying Picture, 20
The z+1/z Substitution, 38

Vieta's Formulas, 37